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# **MEMORANDUM**

SAMPLED-DATA TECHNIQUES APPLIED TO A DIGITAL  
CONTROLLER FOR AN ALTITUDE AUTOPILOT

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**NATIONAL AERONAUTICS AND  
SPACE ADMINISTRATION**

**WASHINGTON**

June 1959



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NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

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MEMORANDUM 4-14-59A

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SUMMARY

Sampled-data theory, using the  $Z$  transformation, is applied to the design of a digital controller for an aircraft-altitude autopilot. Particular attention is focused on the sensitivity of the design to parameter variations and the abruptness of the response, that is, the normal acceleration required to carry out a transient maneuver. Consideration of these two characteristics of the system has shown that the finite settling time design method produces an unacceptable system, primarily because of the high sensitivity of the response to parameter variations, although abruptness can be controlled by increasing the sampling period. Also demonstrated is the importance of having well-damped poles or zeros if cancellation is attempted in the design methods.

A different method of smoothing the response and obtaining a design which is not excessively sensitive is proposed, and examples are carried through to demonstrate the validity of the procedure. This method is based on design concepts of continuous systems, and it is shown that if no pole-zero cancellations are allowed in the design, one can obtain a response which is not too abrupt, is relatively insensitive to parameter variations, and is not sensitive to practical limits on control-surface rate. This particular design also has the simplest possible pulse transfer function for the digital controller.

Simulation techniques and root loci are used for the verification of the design philosophy.

INTRODUCTION

The design of sampled-data systems, the theory of which is applicable to feedback control systems utilizing a digital computer, is a relatively new field but is receiving considerable attention in those applications in which the basic information is received as pulses or in the form of numbers. This may be due to greater accuracy requirements, since digital transducers can be made more accurate than analog transducers, or to the fact that the actual measurement device, for example a radar, yields a sampled signal. Another application in which digital computers in real

time control systems are receiving attention is where a large amount of flexibility of changing computation procedures is required (e.g., an interceptor aircraft having a fire control system, an automatic landing system, altitude and direction hold systems, etc.). In these applications the digital computer offers a possibility of weight reduction and improved reliability, both highly desirable features.

The sampled-data theory<sup>1</sup> has progressed to the point where it is possible to design a digital-controller pulse transfer function so as to obtain an over-all system pulse transfer function which will meet certain specifications, such as being stable and physically realizable, having a minimum settling time in response to a given input, having a ripple-free response after a short transient, etc. A survey of the literature, however, has indicated a scarcity of information on the sensitivity of a system to parameter variations or how to control, in the design process, the abruptness of the response. References 2 and 3 touch upon the subject of abruptness and show one method of smoothing the response by proper design of the characteristic equation. Another shortcoming in the literature has been that the examples are restricted to plants of third order or less.

It is the purpose of this paper to show, first, that considerations of the sensitivity of the response to parameter variations and abruptness of the response to transient inputs preclude the use of finite settling time design (i.e., one in which the error is reduced to zero in a specified number of sampling instants); second, to show that continuous system design concepts can be used to select the dominant poles of the closed-loop pulse transfer function so as to achieve a desired transient response; and third, to present results of a study of the design for a fourth-order plant.

In order to obtain some indication of the practical limitations imposed on the response of the system, the "plant" is taken to be an aircraft and the system to be one that controls the aircraft's altitude. The abruptness and sensitivity of the response to a transient maneuver will be judged in this application by the magnitude of the maximum load factor and the change in the stability of the system with parameter variations.

In order to obtain a system response that will be acceptable to a variety of inputs and flight conditions the theory of design used here is different from that normally employed to "soften" the response.

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<sup>1</sup>The theory and design methods contained in this report were derived to a large extent from a course taught at Stanford University by G. F. Franklin. Reference 1 by J. R. Ragazzini and G. F. Franklin contains an extensive bibliography of the available literature on sampled-data theory.

## NOTATION

$a_z$	normal acceleration, ft/sec <sup>2</sup>
$\bar{c}$	mean aerodynamic chord, ft
$C_m$	pitching-moment coefficient, $\frac{\text{pitching moment}}{(1/2)\rho V^2 S \bar{c}}$
$C_z$	vertical-force coefficient, $\frac{\text{vertical force}}{(1/2)\rho V^2 S \bar{c}}$
$e$	2.7183
$h$	altitude, ft
$\Delta h$	change in altitude, ft
$I_y$	moment of inertia, lb-ft/sec <sup>2</sup>
$j$	$\sqrt{-1}$
$m$	mass, lb sec <sup>2</sup> /ft
$M_\alpha, M_q$	$\frac{1}{2} \frac{\rho V^2 S \bar{c}}{I_y} \left( \frac{\partial C_m}{\partial \alpha}, \frac{\partial C_m}{\partial q \bar{c}/2V} \frac{\bar{c}}{2V} \right)$
$M_{\dot{\alpha}}, M_\delta$	$\frac{1}{2} \frac{\rho V^2 S \bar{c}}{I_y} \left( \frac{\partial C_m}{\partial \dot{\alpha} \bar{c}/2V} \frac{\bar{c}}{2V}, \frac{\partial C_m}{\partial \delta} \right)$
$q$	pitching velocity, radians/sec
$s$	Laplace operator
$S$	wing area, ft <sup>2</sup>
$T$	sampling period, sec
$V$	velocity, ft/sec
$z$	$e^{sT} = Z$ transform operator
$Z_\alpha, Z_\delta$	$\frac{1}{2} \frac{\rho V S}{m} \left( \frac{\partial C_z}{\partial \alpha}, \frac{\partial C_z}{\partial \delta} \right)$
$Z[f(t)]$	$Z$ transform of $f(t)$
$\alpha$	angle of attack, radians

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$\gamma$	flight path angle, radians
$\delta$	control-surface deflection, radians
$\zeta$	damping ratio
$\theta$	pitch angle, radians
$\pi$	3.14159
$\rho$	density, lb sec <sup>2</sup> /ft <sup>4</sup>
$\omega_n$	natural frequency, radians/sec
$F(z^{-1})$	finite polynomial in $z^{-1}$ of form $\sum_{i=0}^n a_i z^{-i}$

Superscript

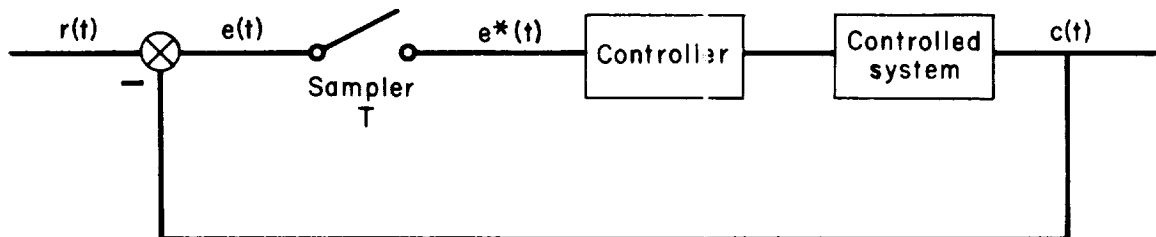
\* sampled version of a function of time

## FUNDAMENTAL CONCEPTS

The theory of sampled-data systems is covered in considerable detail in references 1 and 4. However, it is desirable to include herein certain of the fundamental concepts of sampled-data systems so that the application of the theory to the example chosen will be understandable by those familiar with the design and analysis of linear feedback systems.

### Elements of a Sampled-Data System

The basic elements of a simple sampled-data system are shown in sketch (a). Here the continuous error signal  $e(t)$  is measured at regular intervals,  $T$  seconds apart, by the sampler. The output of the



Sketch (a)



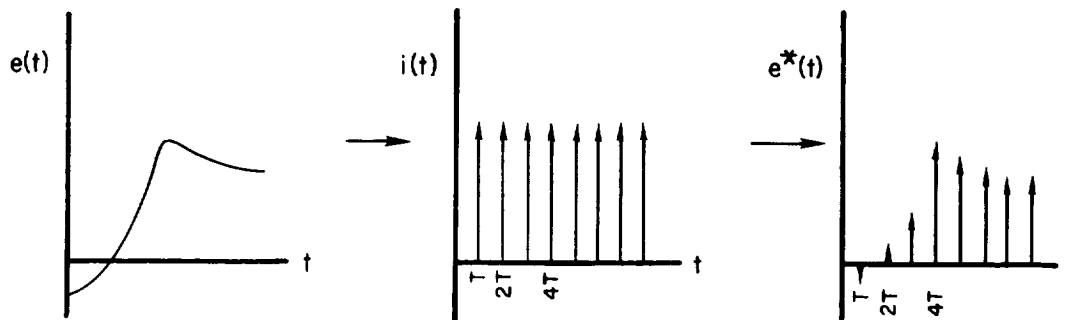
sampler consists of a train of pulses whose amplitude (or area) represents the value of the input at each sampling instant. This pulse train  $e^*(t)$  is modified by the controller to provide stability and other characteristics to the complete system as well as smoothing of the sampled data. The controlled-system characteristics are presumed known and in this report will be represented by a transfer function relating altitude to controller output.

### Analytical Description of the Sampler

The output of the sampler shown in sketch (a) can be written as

$$e(t)i(t) = e^*(t) \quad (1)$$

where  $i(t)$  is a train of unit impulses occurring every  $T$  seconds. This operation is illustrated in sketch (b)



Sketch (b)

Now using an infinite series expression for  $i(t)$  the sampler output becomes

$$e^*(t) = e(t) \sum_{n=0}^{\infty} u_0(t - nT) \quad (2)$$

where  $u_0$  is the unit impulse function. Since the unit impulse function is zero except at time  $nT$ , this equation can be rewritten as

$$e^*(t) = \sum_{n=0}^{\infty} e(nT)u_0(t - nT) \quad (3)$$

where  $e(nT)$  is the value of the input when  $t = nT$ . The Laplace transform of this equation is

$$E^*(s) = \sum_{n=0}^{\infty} e(nT)e^{-nTs} \quad (4)$$

A second method of specifying  $E^*(s)$  is by expanding

$$\sum_{-\infty}^{+\infty} u_0(t - nT) = i(t)$$

in a Fourier series which gives

$$i(t) = \frac{1}{T} \sum_{-\infty}^{+\infty} e^{j2\pi nt/T} \quad (5)$$

since the Fourier series coefficients are constant and all equal to  $1/T$ ; therefore,

$$e^*(t) = e(t) \frac{1}{T} \sum_{-\infty}^{+\infty} e^{j2\pi nt/T} \quad (6)$$

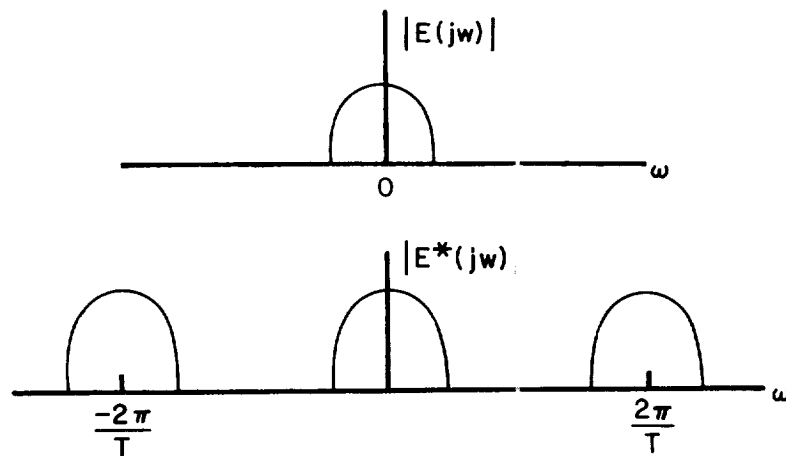
By theorem (see ref. 5)

$$L[f(t)e^{j\omega t}] = F(s - j\omega) \quad (7)$$

then,

$$E^*(s) = \frac{1}{T} \sum_{-\infty}^{+\infty} E\left(s - \frac{2\pi nj}{T}\right) \quad (8)$$

This formula shows that  $E^*(s)$  is a periodic function repeating itself every  $2\pi/T$  radians per second as illustrated in sketch (c).



Sketch (c)

## The Z Transform

The Z transform of a function of time (defined only at sampling instants) can be found by substituting  $z = e^{sT}$  in equation (4). Thus

$$Z[e^*(t)] = E(z) = \sum_{n=0}^{\infty} e(nT)z^{-n} \quad (9)$$

The infinite summation can be found in closed form for all applications considered here. To find the closed form one can either sum the series (9) or use the complex convolution integral

$$E(z) = E^*(s) \bigg|_{z=e^{sT}} = \frac{1}{2\pi j} \int_{c-j\infty}^{c+j\infty} E(\lambda) \frac{d\lambda}{1 - e^{-T(s-\lambda)}} \bigg|_{z=e^{sT}} \quad (10)$$

Tables have been prepared for a large number of the useful Z transforms (refs. 1 and 4); thus, in general it is not necessary to use complex integration. The Z transform can relate the sampled input of a system to the sampled output. When it is used in this manner it is commonly referred to as a "pulse" transfer function.

To obtain  $e(nT)$  from a closed Z transform there are three courses that can be taken

1. Refer to tables (refs. 1 and 4).
2. Expand  $E(z)$  in powers of  $z^{-1}$  by long division.
3. Use the formula  $e(nT) = \frac{1}{2\pi j} \oint_{\Gamma} E(z)z^{n-1}dz$  where  $\Gamma$  is the unit circle.

The initial and final value of a pulse sequence which results from inversion of a pulse transfer function can be determined readily from the following theorems:

$$\text{Initial value theorem} \quad f(0) = \lim_{z \rightarrow \infty} F(z) \quad (11)$$

$$\text{Final value theorem} \quad f(\infty) = \lim_{z \rightarrow 1} (1 - z^{-1})F(z) \quad (12)$$

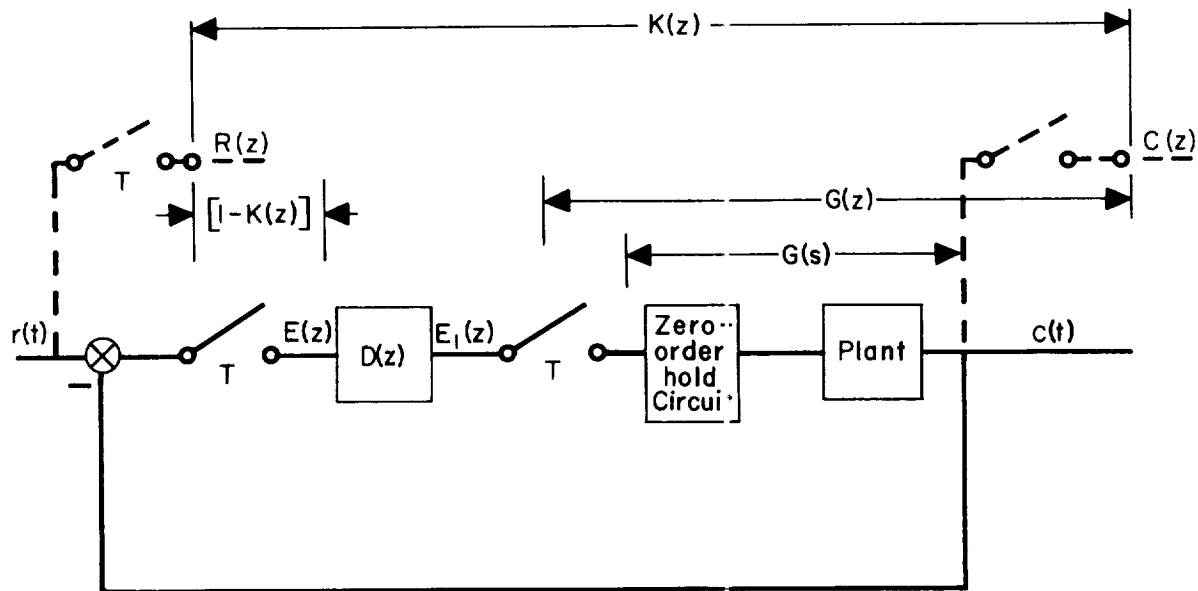
A further important point about the Z transform is the stability of a given pulse transfer function. Since  $z = e^{sT}$  it can be seen that the left half of the s plane maps into a unit circle in the z plane. Thus, if all poles of a pulse transfer function are located inside this unit circle the pulse transfer function is stable.

In general, for a sampled-data system of the type illustrated in sketch (a), that is a sampling of the error signal, the  $Z$  transform can be used similarly to the Laplace transform for a continuous system. Root-locus methods of analysis can be used in the  $z$  plane; however, they are difficult to apply since the region of interest is that inside the unit circle. The drawing of the root locus becomes relatively tedious since asymptotic behavior provides no help in giving a rough idea of the position of the loci as is the case in the  $s$  plane. It should also be recognized that the  $Z$  transform design specifies only what happens at sampling instants. This is not a serious drawback, however, for in order to determine system behavior it is always possible to test the designed system by simulation techniques. Also, from the transfer function of the continuous elements of the system, a relatively accurate idea of the system behavior between sampling instants can be determined.

Because of the drawbacks of root-locus techniques in the  $z$  plane and for other reasons, the design of sampled-data systems is probably best carried out by a different method. This method, which is explained in the next section, could be applied to continuous system design as well; however, to the authors' knowledge, it is not in widespread use.

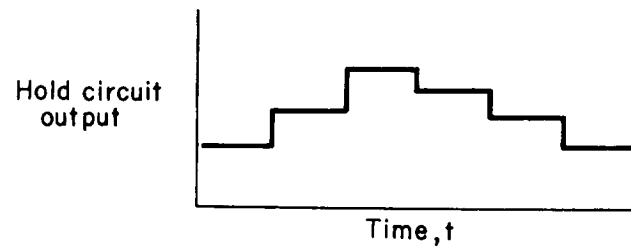
#### Design Criteria For Digital-Controller Pulse Transfer Function

The procedures for the design of the digital controller  $D(z)$  are outlined with reference to the simplified block diagram shown in sketch (d). In this sketch  $D(z)$  represents the pulse transfer function of the digital controller. The zero-order hold circuit constructs a continuous signal



Sketch (d)

from a sampled signal as shown in sketch (e). The transfer function for this operation is  $(1 - e^{-sT})/s$ . The block labeled "plant" represents the fixed elements of the system.



Sketch (e)

The following equations are developed from sketch (d):

$$K(z) = \frac{C(z)}{R(z)} \quad (13)$$

$$1 - K(z) = \frac{E(z)}{R(z)} \quad (14)$$

$$\frac{K(z)}{1 - K(z)} = \frac{C(z)}{E(z)} = D(z)G(z) \quad (15)$$

therefore

$$D(z) = \frac{K(z)}{1 - K(z)} \frac{1}{G(z)} \quad (16)$$

Equation (16) is the fundamental design equation. Certain mathematical constraints must be put on this equation in order to arrive at a stable system which has the desired characteristics. These constraints and the reasons for them are as follows:

#### I. Stability

A.  $K(z)$  must be of the form

$$[(1 - a_1 z^{-1})(1 - a_2 z^{-1}) \dots](\beta_0 + \beta_1 z^{-1} + \dots) \quad (17)$$

where the  $\beta$ 's are undetermined coefficients and  $z = a_1$ ,  $z = a_2$ ,  $\dots$  are the zeros of  $G(z)$  on or outside the unit circle (on or outside the stable region in the  $z$  plane). The reason for this constraint is obvious from equation (16). If it is not satisfied  $D(z)$  will have a pole where  $G(z)$  has a zero, and consequently some of the poles of  $D(z)$  will be outside the stable region causing the over-all system to be unstable since perfect cancellation can never be expected.

B.  $1 - K(z)$  must be of the form

$$[(1 - b_1 z^{-1})(1 - b_2 z^{-1}) \dots](\gamma_0 + \gamma_1 z^{-1} + \dots) \quad (18)$$

where the  $\gamma$ 's are undetermined coefficients and  $z = b_1$ ,  $z = b_2$ ,  $\dots$  are the poles of  $G(z)$  on or outside the unit circle in the  $z$  plane. The reason for this constraint is also obvious from equation (16), for if it were not met, zeros of  $D(z)$  would be imperfectly canceling poles of  $G(z)$  outside the stable region, a fact which would cause instability in the over-all system.

## II. Zero Steady-State Error to an Input of the Form $r(t) = t^n$

The  $Z$  transform of  $t^n$  is of the form  $[F(z^{-1})]/[(1 - z^{-1})^{n+1}]$ . If zero error to such an input is desired (at sampling instants) then  $1 - K(z)$  must be of the form

$$(1 - z^{-1})^{n+1}[F_1(z^{-1})] \quad (19)$$

where  $F_1(z^{-1})$  is a polynomial in  $z^{-1}$  satisfying other constraints. This can be seen by applying the final value theorem (12) to equation (14).

## III. Transient Performance

### A. Finite settling time

#### (1) Minimum

If only I and II are satisfied then  $K(z)$  will be the lowest order polynomial in  $z^{-1}$ . This results in a minimum finite settling time since reference to equation (9) shows that  $e(nT)$  will be zero (to a unit pulse input  $r(t)$ ) after  $T$  times the order of  $K(z)$  seconds.

#### (2) Zero ripple

Zero ripple by definition means that for an input of form  $r(t) = t^n$  the output must be of the form  $t^n$ .

In other words, the output must follow the input between sampling instants. With reference to sketch (d) this can only be accomplished if  $e_1(nT)$  is zero or a constant in the steady state, for otherwise the output of the zero-order hold circuit would be stepping from one value to another, which would cause ripple.

$$E_1(z) = \frac{C(z)}{G(z)} = R(z) \frac{K(z)}{G(z)} \quad (20)$$

For  $e_1(nT)$  equal to a constant in the steady state, equation (20) must be of the form

$$\frac{F(z^{-1})}{1 - z^{-1}}$$

as can be seen from equation (12). Thus, for zero ripple all the zeros of  $G(z)$  must be contained in  $K(z)$ . This is not the only requirement. In addition if

$$R(z) = \frac{F(z^{-1})}{(1 - z^{-1})^{n+1}}$$

then  $G(z)$  must have an  $n$ th-order pole at  $z = 1$  to cancel all but one of the poles of  $R(z)$ . An even simpler way of assuring that zero ripple is possible is to count the number of poles of  $G(s)$  at  $s = 0$ . If  $r(t) = t^n$  then the number of poles of  $G(s)$  at  $s = 0$  must be greater than or equal to  $n+1$ .

#### B. Smoothing

As is described in reference 1 it is possible to add a denominator<sup>2</sup> to  $K(z)$  which has been determined by the preceding results so that  $K(z)$  is of the form

$$K(z) = \frac{F(z^{-1})}{(1 + c_1 z^{-1} + c_2 z^{-2} + \dots)} \quad (21)$$

The addition of this denominator changes the system such that finite settling time is no longer obtained. It is mentioned in reference 1 that the choices of  $c$ 's can be made for

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<sup>2</sup>Note that in reality we are not necessarily increasing the order of the characteristic equation. What is really done is to change the characteristic equation from one with all roots at  $z = 0$  to one which allows some of the roots to be at other positions inside the unit circle in the  $z$  plane.

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(1) Smoothing the transient response so that it is not as abrupt.

(2) Optimizing the response in the presence of noise.

This report illustrates that if considerations of sensitivity and abruptness of response are important, then one should always design the system so that  $K(z)$  has a denominator. It is shown in a later section of this report that a choice of  $c$ 's in equation (21) which will satisfy abruptness of response requirements can be determined from continuous system design theorems.

#### DESCRIPTION OF PROBLEM

The problem chosen as an illustrative example is an altitude command-type autopilot. In order to reduce the problem to block diagram form it is necessary to derive the aerodynamic equations and put them into transfer function form. In order to do this certain assumptions must be made as follows:

- (a) The aircraft does not roll.
- (b) The velocity is a constant.
- (c) The altitude is approximately constant.
- (d) Small angle approximations are valid for  $\theta$  and  $\gamma$ .
- (e) The aircraft is initially in trim flight.
- (f) The aerodynamic coefficients are constant.

With these assumptions the following equations are valid

$$-q + \dot{\alpha} = -\dot{\gamma} = Z_{\alpha}\alpha + Z_{\delta}\delta \quad (22)$$

$$\dot{q} = M_{\alpha}\alpha + M_q q + M_{\delta}\delta + M_{\dot{\alpha}}\dot{\alpha} \quad (23)$$

$q = \dot{\theta}$       pitching velocity

$V\gamma = a_z$       normal acceleration

$V\dot{\gamma} = \dot{h}$       rate of change of altitude

from which the transfer functions given below can be derived:



$$\frac{q}{\delta} = \frac{(M_{\delta} + M_{\dot{\alpha}}Z_{\delta})s + (M_{\alpha}Z_{\delta} - M_{\delta}Z_{\alpha})}{s^2 - (Z_{\alpha} + M_q + M_{\dot{\alpha}})s + (M_qZ_{\alpha} - M_{\alpha})} \quad (24)$$

$$\frac{V\dot{\gamma}}{\delta} = \frac{a_z}{\delta} = V \frac{(-Z_{\delta})s^2 + (M_q + M_{\dot{\alpha}})Z_{\delta}s - (M_{\delta}Z_{\alpha} - M_{\alpha}Z_{\delta})}{s^2 - (Z_{\alpha} + M_q + M_{\dot{\alpha}})s + (M_qZ_{\alpha} - M_{\alpha})} \quad (25)$$

As a representative aircraft an interceptor at a cruising velocity of 695 feet per second, 30,000 feet altitude was chosen. The aerodynamic constants are tabulated below

$$\begin{aligned} M_{\delta} &= -16.04 & M_q &= -0.489 \\ M_{\dot{\alpha}} &= -0.1630 & M_{\alpha} &= -6.737 \\ Z_{\alpha} &= -0.6716 & Z_{\delta} &= -0.1205 \\ V &= +695 \text{ ft/sec} \end{aligned}$$

With these values the transfer functions (24) and (25) become

$$\frac{q}{\delta} = \frac{-16.02s - 9.961}{s^2 + 1.324s + 7.065} \quad (26)$$

$$\frac{V\dot{\gamma}}{\delta} = \frac{a_z}{\delta} = \frac{83.75s^2 + 54.61s - 6923}{s^2 + 1.324s + 7.065} \quad (27)$$

The block diagram of the system is as shown in figure 1. Two cases designed for comparative purposes are as follows:

$$\text{Case I} \quad K_q = 0$$

$$\text{Case II} \quad K_q = -0.18$$

These values of  $K_q$  result in the transfer functions between  $M$ , the output of the hold circuit, and  $a_z$  of

$$\left. \begin{aligned} \text{Case I} \quad \frac{\delta}{M} &= 1.0 \quad \therefore \quad \frac{a_z}{M} = \frac{a_z}{\delta} \\ \text{Case II} \quad \frac{a_z}{M} &= \frac{83.75s^2 + 54.61s - 6923}{s^2 + 4.206s + 8.858} \end{aligned} \right\} \quad (28)$$

## APPLICATION OF DESIGN CRITERIA TO ALTITUDE AUTOPILOT

Theoretically, the only constraint which must be imposed upon the design of a digital controller is that the system be stable. As shown in the previous section a minimum finite settling time system would result from this constraint alone. Since it is desirable to have zero error continuously one must also impose the zero ripple constraint. Practical considerations, however, such as abruptness of the response to step inputs, smoothing of noisy inputs, and sensitivity to parameter changes may require that additional constraints on the design be imposed. It is desirable, therefore, to investigate the different designs which can be made and study their characteristics by means of simulation.

## Choice of Sampling Period

There seems to be no theoretical method of choosing the optimum sampling period for a closed-loop system. If the input command were band limited to a frequency  $\omega$ , then the sampling theorem states that if  $1/2\omega = T$ , all information can be reconstructed from the samples. This theorem does not apply here since the system is completely satisfactory only if zero error to all possible inputs is maintained. This dilemma is not studied here. The choice of sampling period is based on knowledge of the control of an aircraft by a human pilot. Previous studies indicated that a period of 0.25 second is about the longest that should be chosen. For exemplary purposes a period of 1 second is also chosen in order that effects of sampling period in the design can be demonstrated. Practical considerations probably would force the choice of period to be the shorter of the two, principally because a 1 second sampling period would cause a rough flight for the pilot. The 0.25 period or 4 cps sampling rate appears to be high enough that the jerkiness of the hold circuit output during transient inputs would be quite well filtered by the control-surface servo and aerodynamic lags.

## Finite Settling Time Design With Zero Ripple

Finite settling time design, Case I,  $K_d = 0$ ,  $T = 0.25$  second. - The steps involved in the design are shown for this example for the benefit of those unfamiliar with digital-controller design methods. The first step is the determination of  $G(z)$

$$G(s) = \left( \frac{1 - e^{-sT}}{s} \right) \left( \frac{83.75s^2 + 54.61s - 6923}{s^2 + 1.324s + 7.065} \right) \left( \frac{1}{s^2} \right) \quad (29)$$

Transfer function (29) is expanded in partial fractions

$$G(s) = (1 - e^{-sT}) \left( -\frac{979.8}{s^3} + \frac{191.3}{s^2} + \frac{114.7}{s} - \frac{114.7s + 343.1}{s^2 + 1.324s + 7.065} \right) \quad (30)$$

The individual terms are converted to  $Z$  transforms, using the tables of reference 1 or 4, and then recombined to give

$$G(z) = \frac{1.360z^3 - 12.67z^2 - 11.96z + 1.074}{(z - 1)^2(z^2 - 1.356z + 0.7182)} \quad (31)$$

$$= \frac{1.360z^{-1} - 12.67z^{-2} - 11.96z^{-3} + 1.074z^{-4}}{(1 - z^{-1})^2(1 - 1.356z^{-1} + 0.7182z^{-2})} \quad (32)$$

The next steps are to determine  $K(z)$  and  $1 - K(z)$  such that stability and zero ripple constraints are satisfied.

$$\begin{aligned} K(z) &= (\text{Numerator of } G(z) \text{ in powers of } z^{-1})(\beta_0 + \beta_1 z^{-1}) \\ &= (1.360z^{-1} - 12.67z^{-2} - 11.96z^{-3} + 1.074z^{-4})(\beta_0 + \beta_1 z^{-1}) \end{aligned} \quad (33)$$

$$\begin{aligned} 1 - K(z) &= \left[ \begin{array}{c} \text{Poles of } G(z) \text{ on or outside unit} \\ \text{circle in powers of } z^{-1} \end{array} \right] (\gamma_0 + \gamma_1 z^{-1} + \gamma_2 z^{-2} + \gamma_3 z^{-3}) \\ &= (1 - 2z^{-1} + z^{-2})(\gamma_0 + \gamma_1 z^{-1} + \gamma_2 z^{-2} + \gamma_3 z^{-3}) \end{aligned} \quad (34)$$

The coefficients of equal powers of  $z^{-1}$  in the expression  $1 - K(z)$  as obtained from equations (33) and (34) are equated. Enough undetermined coefficients must be provided to give a sufficient number of simultaneous equations in  $\gamma$ 's and  $\beta$ 's to allow their solution. For this example, the solution of the simultaneous equations gives

$$\begin{aligned} \gamma_0 &= 1.000 & \beta_0 &= -0.1579 \\ \gamma_1 &= 2.215 & \beta_1 &= 0.1128 \\ \gamma_2 &= 1.276 \\ \gamma_3 &= -0.1211 \end{aligned}$$

Then

$$K(z) = (1.360z^{-1} - 12.67z^{-2} - 11.96z^{-3} + 1.074z^{-4})(-0.1579 + 0.1128z^{-1}) \quad (35)$$

$$1 - K(z) = (1 - z^{-1})^2(1 + 2.215z^{-1} + 1.276z^{-2} - 0.1211z^{-3}) \quad (36)$$

The last step is the substitution of equations (35), (36), and (32) into equation (16) to give

$$D(z) = \frac{-0.1579(1 - 0.7146z^{-1})(1 - 1.356z^{-1} + 0.7182z^{-2})}{1 + 2.215z^{-1} + 1.276z^{-2} - 0.1211z^{-3}} \quad (37)$$

The design at this stage is complete. This system as designed, however, may be unsatisfactory because of sensitivity to parameter changes and abruptness of the response. An analog computer simulation is one of the best ways of analyzing such a system. This method is used here. Appendix A shows one procedure by which a digital computer can be simulated on an analog computer. This method is derived in reference 6. Derived, also in appendix A, is the detailed computer diagram for this example. Appendix B contains a description of the electronic sample-and-hold circuit used for simulation purposes in this investigation.

The results of the simulation studies are summarized in figure 2. There are two important things to note in figure 2. The first is that the system does not have a finite settling time. This is obvious from figures 2(a) and (b) in which it can be seen that both the step response and the ramp response contain an oscillatory mode which damps exponentially. This appears to be due to the extreme sensitivity of the system which makes it impossible to simulate the system even with highly accurate analog computing equipment. The second item of importance to note is that for a step input of 10 feet, figure 2(a), a peak acceleration of approximately 31 g's is required during the transient maneuver. This large peak in acceleration is a result of the linear analysis. In an actual aircraft, control-surface position and rate limits as well as aerodynamic nonlinearities would prevent such an excessive peak. One cannot state what the exact performance of the finite settling time design would be with the actual nonlinearities; however, the introduction of a high-performance servo in the simulation resulted in an instability of the system which indicates that in all probability nonlinearities would also cause system instability.

Figure 2(c) illustrates the effects of a 10-percent increase and decrease in system gain. These results plus the fact that the simulated system does not have finite settling time indicate that the design is quite sensitive to parameter changes.

Figure 3 is a root-locus plot showing effects of gain on theoretical pole locations of a closed-loop system. The system is noted to be very sensitive to gain changes around the designed gain which places the closed-loop poles at the origin.

Figures 4(a) and (b) are root-locus plots showing effects of altitude on the closed-loop pole-zero locations. Note that a drop in altitude changes all the aerodynamic coefficients because of the consequent change in air density. Again it is seen that the pole locations which are at the origin for 30,000 feet move a considerable distance (two almost becoming unstable) for 27,000 feet.

In summary, then, for two reasons - (a) relatively high sensitivity to parameter changes and (b) response which is much too abrupt - the finite settling design method does not result in a satisfactory closed-loop performance for this particular case.

Finite settling time design, Case II,  $K_d = -0.18$ ,  $T = 0.25$  second.-

The results of the previous system studied showed that the theoretical finite settling time design could not be simulated. Figures 3 and 4 illustrate the relatively high sensitivity of the system; however, this is not necessarily the only reason that the simulated system did not agree with theory. Another reason might be related to the fact that zeros of  $D(z)$  theoretically must cancel poles of  $G(z)$ . Because the two canceled poles have a relatively low damping ratio ( $\xi = 0.258$ ), it is possible that imperfect cancellation, in addition to the high sensitivity previously noted, could be an important factor in the disagreement. The artificial damping added in Case II increases the damping ratio of the canceled poles to 0.707, thus making it possible to study the effect of damping on the ability to simulate a finite settling time design.

For this example, the use of the same procedures as those of the previous section gives the following pulse transfer functions:

$$G(z) = \frac{1.019z^3 - 10.21z^2 - 7.527z + 0.7295}{(z - 1)^2(z^2 - 1.022z + 0.3494)} \quad (38)$$

$$K(z) = (1.019z^{-1} - 10.21z^{-2} - 7.527z^{-3} + 0.7295z^{-4})(-0.2154 + 0.1528z^{-1}) \quad (39)$$

$$1 - K(z) = (1 - z^{-1})^2(1 + 2.220z^{-1} + 1.085z^{-2} - 0.1115z^{-3}) \quad (40)$$

$$D(z) = \frac{-0.2154(1 - 0.7096z^{-1})(1 - 1.022z^{-1} + 0.3494z^{-2})}{1 + 2.220z^{-1} + 1.085z^{-2} - 0.1115z^{-3}} \quad (41)$$

The results of the simulated response for this example are summarized in figure 5. It can be noted from both figures 5(a) and 5(b) that the simulated system has a finite settling time, a fact which agrees with the theory. Figure 5(a) shows this system also to be very abrupt, as would be expected, requiring about 33 g's peak for a 10-foot step input. Figure 5(c) illustrates that the response of this system is also quite sensitive to gain variations.

From the results of the simulated systems, Case I and Case II, it can be concluded that if the design method being utilized requires pole-zero cancellation one should be certain the canceled poles of  $G(s)$  are in a well-damped region of the  $s$  plane.

Finite settling time design,  $T = 1.0$  second.- The two cases studied previously are unacceptable for application to an aircraft due to both sensitivity to parameter variations and abruptness of response. One of the questions it is necessary to answer for a sampled-data system is: Do these two parameters vary together or can they be independently controlled? It should be obvious that one way of controlling abruptness of response is by increasing the sampling period. Thus, one can maintain a finite settling time design and reduce the abruptness of response, but the important question is will the sensitivity to parameter changes also be reduced?

An alternative procedure which could be utilized is to keep the sampling period the same but specify, as another constraint, the time response at sampling instants to be smooth and as slow as desired for as many sampling instants as is necessary. If one maintains a finite settling time design in this manner the result will be that the order of the system is increased by one for each sampling instant specified; thus, the digital computer will become more and more complicated. This second method will not be illustrated here because of its additional complexity.

To determine the effect of sampling period on abruptness and sensitivity, the two cases are designed for finite settling time period  $T = 1$  second. For Case I,  $K_q = 0$ , the pulse transfer functions are:

$$G(z) = \frac{-162.8z^3 - 1104z^2 - 754.9z - 72.11}{(z-1)^2(z^2 + 0.8704z + 0.2662)} \quad (42)$$

$$K(z) = (-162.8z^{-1} - 1104z^{-2} - 754.9z^{-3} - 72.11z^{-4})(-0.001601 + 0.001123z^{-1}) \quad (43)$$

$$1 - K(z) = (1 - 2z^{-1} + z^{-2})(1 + 1.739z^{-1} + 0.8947z^{-2} + 0.08106z^{-3}) \quad (44)$$

$$D(z) = \frac{-0.001601(1 - 0.7016z^{-1})(1 + 0.8704z^{-1} + 0.2662z^{-2})}{1 + 1.739z^{-1} + 0.8947z^{-2} + 0.08106z^{-3}} \quad (45)$$

This system was simulated by the method of appendix A and typical transient responses are summarized in figure 6. Again it may be noted that in the simulation, finite settling time was not achieved. The abruptness of the response has been considerably reduced, that is, peak acceleration is about 0.87 g's for a 10-foot step. Figure 6(c) shows the response to be somewhat sensitive to gain changes; however, the best indication of a sensitivity problem is the inability to simulate a theoretically finite settling time design. The reason for this could be the attempt to cancel poles of  $G(z)$  which are in relatively low damped regions as was the case for  $T = 0.25$  second. This particular system is too oscillatory to be useful so we shall now consider Case II where canceled poles are more heavily damped. In this case the pulse

transfer functions are:

$$G(z) = \frac{-110.3z^{-1} - 572.1z^{-2} - 200.6z^{-3} - 7.468z^{-4}}{(1 - z^{-1})^2(1 + 0.1247z^{-1} + 0.01491z^{-2})} \quad (46)$$

$$K(z) = (-110.3z^{-1} - 572.1z^{-2} - 200.6z^{-3} - 7.468z^{-4})(-0.003501 + 0.002378z^{-1}) \quad (47)$$

$$1 - K(z) = (1 - z^{-1})^2(1 + 1.614z^{-1} + 0.4866z^{-2} + 0.01780z^{-3}) \quad (48)$$

$$D(z) = \frac{-0.003501(1 - 0.6793z^{-1})(1 + 0.1247z^{-1} + 0.01491z^{-2})}{1 + 1.614z^{-1} + 0.4866z^{-2} + 0.01780z^{-3}} \quad (49)$$

The results of this simulation are summarized in figure 7. Figures 7(a) and (b) show that finite settling time was achieved for this case, and illustrates again that if poles of  $G(z)$  are to be canceled by zeros of  $D(z)$  then these poles must be heavily damped. Figure 7(a) also illustrates, as was anticipated, that the abruptness of response has been reduced considerably being about 1.30 g's for a 10-foot step input. Figure 7(c) also gives conclusive evidence that the finite settling time design is quite sensitive to gain variations even though the abruptness of response has been reduced. Further evidence of this sensitivity problem is illustrated in figure 8 which is a root-locus plot of Case II for open-loop gain variations.

The general summary of the results would tend to confirm the fact that a finite settling time design procedure will always result in a sensitive system and, thus, if it is to be used for any practical applications, the plant must have a transfer function free of nonlinearities and with coefficients that are nonvariant. An intuitive reason for this is illustrated by the root-locus plot of figure 8. We see that a finite settling time design requires all poles of the closed-loop system, with the exception of those being canceled, to be at the origin in the  $z$  plane. It can be seen the origin is a position of extreme sensitivity since open-loop gain changes of  $\pm 5$  percent cause the poles to move a considerable distance; whereas, an additional change of  $\pm 5$  percent causes only the small motion indicated in the figure. The alternative procedure for reducing the abruptness of the response, previously mentioned, would place an even greater number of poles at the origin. From the results shown in figure 8, one would believe that placing more poles at the origin would increase the sensitivity rather than decrease it.

It must be concluded from these standpoints that some of the poles of the closed-loop system should be located in other positions than the origin inside the unit circle of the  $z$  plane if we are to obtain satisfactory performance. The next section illustrates a method for choosing this location based on continuous-system analogy.

### Considerations Involved in Adding a Denominator to $K(z)$

The two previous systems studied indicate that it would be desirable to reduce the sensitivity to parameter variations and smooth the transient response. It is mentioned in references 2 and 3 that the addition of a denominator (sometimes referred to as a staleness factor if the denominator is first order) will smooth the transient response. It will be shown here that if the denominator is properly chosen, the sensitivity to parameter changes is reduced.

There are two possible methods for selecting a denominator. The first is a simple trial and error process which can be very tedious. The second is to base the sampled-data system response on what might be reasonable for a continuous system of the same type. By means of the latter method, experience gained from similar continuous system designs can be used to select the proper location of the dominant modes in the  $s$  plane which are known to give satisfactory performance. These modes or location of poles can then be transferred to the  $z$  plane and the design carried through using the added denominator.

Consider the continuous system shown in the block diagram of figure 9. The problem is to design the network  $D(s)$  so that the over-all closed-loop response  $h_0/h_1$  will be satisfactory. Figure 10 shows the pole and zero locations of  $G(s)$  for the two cases. It is obvious that for Case I the complex poles of the aircraft are insufficiently damped. For this case,  $D(s)$  must either (1) have complex zeros to attract the aircraft poles to a more favorable position or (2) cancel the poles with zeros and place new poles in a more favorable position. The second choice can never be used for an aircraft since (1) poles and zeros are never known very accurately, (2) their positions shift with Mach number and altitude, and (3) gust inputs would excite the oscillatory mode since it is not canceled for inputs other than those from the control surface.

Figure 10(b) illustrates that the effect of adding an inner-loop pitch-rate feedback (fig. 1) is to shift the poles to a more favorable position. It should be noted that aircraft automatic control systems almost always utilize an inner-loop autopilot of either the combined normal acceleration and pitch-rate feedback type or the simple pitch-rate feedback illustrated in this example. The addition of normal acceleration as a feedback along with pitch rate allows both the natural frequency and damping of the aircraft modes to be shifted substantially from the basic airframe oscillatory mode.

It should be noted that  $D(s)$  for the continuous system (Case II) could be a simple lead network provided the zero, pole, and gain of the network were chosen so that the complex poles did not shift to an unfavorable position for the closed-loop performance. Case II is a much more satisfactory system for the continuous system since the network can be simpler. It will therefore be used for the following sampled-data system studies.



For Case II then, what might be a reasonable selection of the dominant second-order mode of the continuous system? This question cannot be answered without specifying the task the system is required to perform. Since this report is dealing principally with the application of sampled-data design techniques, the only concern will be that the closed-loop system be relatively insensitive to parameter changes. The dominant mode of the continuous system will be the closed-loop location of the two poles which are at  $s = 0$  open loop. Figure 11 illustrates a root-locus plot for one location of the zero and pole of a simple lead network for  $D(s)$ . Note that the zeros of  $G(s)$  are omitted since their effects on the loci near the origin are negligible, except that the gain must be negative for stability. It should be noted that there is not much change in the aircraft oscillatory mode if the gain is varied from 0 to 17. A nominal operating gain for the system was chosen as 17 which corresponds to the dominant mode at  $\omega_n = 0.5$  radian per second and  $\zeta = 0.707$ . From the figure it can be seen that the resultant over-all closed-loop transfer function can be approximated by

$$\frac{h_o}{h_i} = \frac{4s + 1}{4s^2 + 2.828s + 1} \quad (50)$$

This approximation is justified because the other three poles and two zeros are a relatively long distance from the dominant real zero and complex-pole locations. It will be assumed here that this transfer function, given by equation (50), satisfactorily performs the task for which the continuous system is being designed.

The dominant mode (denominator) of the sampled-data system closed-loop,  $K(z)$ , will thus be chosen at the  $z$  plane location of the two poles of the denominator of the transfer function given by equation (50). This transforms (for  $T = 0.25$ ) into

$$(z - 0.9126 + 0.8171i)(z - 0.9126 - 0.8171i) = z^2 - 1.825z + 0.8395 \quad (51)$$

Figure 12 illustrates the location of poles and zeros of  $G(z)$  for Case II,  $K_q = -0.18$ ,  $T = 0.25$ . It should be noted that the two open-loop poles at  $z = 1$  will move along the dotted lines as the gain is increased to arrive at the desired location given by equation (51). It should also be noted that  $D(z)$  can be chosen for this case in two different ways. The first method, cancellation permitted, is to cancel the two complex poles of the aircraft by zeros and place other poles so that, at the desired gain, they end up at the origin. The second method, cancellation not permitted, is to choose a single pole for  $D(z)$  and force  $K(z)$  to have at least four poles away from the origin. This second method will actually show that the digital-controller pulse transfer function  $D(z)$  can be simplified by a properly chosen denominator. Both of these methods will be studied in the following sections.

Cancellation permitted, Case II,  $K_q = -0.18$ ,  $T = 0.25$  second.- The method by which a denominator is added to  $K(z)$  is by specifying

$$K(z) = \frac{[F_1(z^{-1}), \text{meeting other constraints}](\beta_0 + \beta_1 z^{-1} + \dots)}{\text{Desired denominator}} \quad (52)$$

and

$$1 - K(z) = \frac{[F_2(z^{-1}), \text{meeting other constraints}](\gamma_0 + \gamma_1 z^{-1} + \dots)}{\text{Desired denominator}} \quad (53)$$

By subtracting equation (52) from unity and equating coefficients of like powers of  $z^{-1}$  to those of equation (53) the  $\beta$ 's and  $\gamma$ 's are uniquely determined. For this example all zeros of  $G(z)$  appear in  $K(z)$  and the two poles at  $z = 1$  appear in  $1 - K(z)$

$$G(z) = \frac{1.019z^{-1} - 10.21z^{-2} - 7.527z^{-3} + 0.7295z^{-4}}{(1 - z^{-1})^2(1 - 1.022z^{-1} + 0.3494z^{-2})} \quad (54)$$

$$K(z) = \frac{(1.019z^{-1} - 10.21z^{-2} - 7.527z^{-3} + 0.7295z^{-4})(\beta_0 + \beta_1 z^{-1})}{(1 - 1.825z^{-1} + 0.8395z^{-2})} \quad (55)$$

$$1 - K(z) = \frac{(1 - z^{-1})^2(\gamma_0 + \gamma_1 z^{-1} + \gamma_2 z^{-2} + \gamma_3 z^{-3})}{(1 - 1.825z^{-1} + 0.8395z^{-2})} \quad (56)$$

The solution for the  $\gamma$ 's and  $\beta$ 's gives

$$\begin{aligned} \gamma_0 &= 1.000 & \beta_0 &= -0.01223 \\ \gamma_1 &= 0.18730 & \beta_1 &= 0.01133 \\ \gamma_2 &= 0.07768 \\ \gamma_3 &= -0.0082657 \end{aligned}$$

Use of equation (16) gives

$$D(z) = \frac{-0.01223(1 - 0.9264z^{-1})(1 - 1.022z^{-1} + 0.3494z^{-2})}{1 + 0.1873z^{-1} + 0.07767z^{-2} - 0.008266z^{-3}} \quad (57)$$

The closed-loop sampled-data system was simulated by the same method as that of previous studies. The results of the simulation are summarized in figure 13. As can be noted in figure 13(a), a 10-foot step now calls for a little over a 1 g maneuver. The response is much slower than for that of the finite settling time design. The abruptness of the control-surface motion at the beginning of the transient causes this peak in normal acceleration and can be traced in part to the design method of canceling the poles of the aircraft. Figure 13(c) illustrates that the system response is not very sensitive to gain variations.

The effects of adding a simulated control-surface servo of transfer function

$$\frac{\delta}{M} = \frac{1}{1 + 0.02s} \quad (58)$$

is shown in figure 13(d) for three values of control-surface rate limiting,  $\delta_{\max}$ . The inclusion of a linear servo alone gave identical response to figure 13(a) so that the large overshoots are directly traceable to effects of control-surface limiting. It is interesting to note that the overshoot is as high as 500 percent for  $\delta_{\max} = 20^\circ$  per second. The unusual fact that the overshoot is higher for  $\delta_{\max} = 20^\circ$  per second than for  $\delta_{\max} = 10^\circ$  per second is due to the time relationship between the actual control-surface motion and the command input during the initial part of the transient. The reason for the overshoot is traceable to the design method of canceling poles with zeros of  $D(z)$ . This might be expected since any nonlinearities such as saturation in the control surface servo velocity make cancellation impossible for a transient input.

Cancellation permitted, Case II,  $K_q = -0.18$ ,  $T = 1.0$  second. - The application of the method previously described for this example gives

$$G(z) = \frac{-110.3z^{-1} - 572.1z^{-2} - 200.6z^{-3} - 7.468z^{-4}}{(1 - z^{-1})^2(1 + 0.1247z^{-1} + 0.01491z^{-2})} \quad (59)$$

$$K(z) = \frac{(-110.3z^{-1} - 572.1z^{-2} - 200.6z^{-3} - 7.468z^{-4})(-0.0009842 + 0.0007864z^{-1})}{1 - 1.320z^{-1} + 0.4966z^{-2}} \quad (60)$$

$$1 - K(z) = \frac{(1 - z^{-1})^2(1 + 0.5710z^{-1} + 0.1622z^{-2} + 0.00587z^{-3})}{1 - 1.320z^{-1} + 0.4966z^{-2}} \quad (61)$$

$$D(z) = \frac{-0.0009842(1 - 0.7990z^{-1})(1 + 0.1247z^{-1} + 0.01491z^{-2})}{1 + 0.5710z^{-1} + 0.1622z^{-2} + 0.00587z^{-3}} \quad (62)$$

Simulation of this system by the method described in appendix A gave the results which are summarized in figure 14. Of particular note in comparing figures 13(a) and 14(a) is that increasing the sampling period does not particularly increase the response time even though there is a considerable reduction in both control-surface deflection and maximum normal acceleration. Figure 14(c) illustrates that the response is not particularly sensitive to parameter changes. The effect of  $\delta$  limiting on this system was studied; however, for a 10-foot step, the called for control-surface motion is so small that control-surface rate limiting produced no noticeable effect.

Cancellation not permitted, Case II,  $K_q = -0.18$ ,  $T = 0.25$  second.-

As was previously mentioned the digital-controller pulse transfer function  $D(z)$  can be simplified by choosing a different characteristic equation. It may be noted in figure 12 that only a single zero and single pole of  $D(z)$ , if their location is properly chosen, will result in a stable design. One of the questions, then, is how to locate the pole and zero. Root-locus methods were tried; however, for a sampled-data system which has as many poles and zeros as this case the root-locus method is very tedious.

An alternative method was used as follows:

$$K(z) = \frac{[F_1(z^{-1})](\beta_0 + \beta_1 z^{-1} + \dots)}{(\text{Desired dominant mode})(1 + c_1 z^{-1} + c_2 z^{-2} + \dots)} \quad (63)$$

$$1 - K(z) = \frac{[F_2(z^{-1})](\gamma_0 + \gamma_1 z^{-1} + \dots)}{(\text{Desired dominant mode})(1 + c_1 z^{-1} + c_2 z^{-2} + \dots)} \quad (64)$$

where  $F_1(z^{-1})$  contains all the zeros of  $G(z)$  and  $F_2(z^{-1})$  contains all the poles of  $G(z)$ . For this example, Case II,  $K_q = -0.18$ ,  $T = 0.25$

$$K(z) = \frac{(1.019z^{-1} - 10.21z^{-2} - 7.527z^{-3} + 0.7295z^{-4})(\beta_0 + \beta_1 z^{-1})}{(1 - 1.825z^{-1} + 0.8395z^{-2})(1 + c_1 z^{-1} + c_2 z^{-2})} \quad (65)$$

$$1 - K(z) = \frac{(1 - z^{-1})^2(1 - 1.022z^{-1} + 0.5494z^{-2})(\gamma_0 + \gamma_1 z^{-1})}{(1 - 1.825z^{-1} + 0.8395z^{-2})(1 + c_1 z^{-1} + c_2 z^{-2})} \quad (66)$$

Solution of simultaneous equations for  $\beta$ 's,  $\gamma$ 's, and  $c$ 's gives

$$\begin{array}{lll} \beta_0 = -0.002763 & \gamma_0 = 1.000 & c_1 = -1.205 \\ \beta_1 = 0.002588 & \gamma_1 = -0.005402 & c_2 = 0.4018 \end{array}$$

By use of equation (16) then

$$D(z) = \frac{-0.002763(1 - 0.9365z^{-1})}{1 - 0.005402z^{-1}} \quad (67)$$

A root-locus plot for this design is illustrated in figure 15. It may be noted that the closed-loop complex poles (due to the aerodynamics) move to a somewhat more highly damped position in the  $z$  plane. Their motion is not large, however, and it would be reasonable to assume that the transient performance will be relatively insensitive to position variations. This conclusion is demonstrated in the simulated responses summarized in figure 16.

Figure 16(d) illustrates that this design which allows no cancellation is insensitive to reasonable values of control-surface limiting. The differences in output are not even large enough to be recorded. The only noticeable difference is in the initial control-surface motion which is completed within one sampling instant and, for the input magnitudes tested, created no noticeable effect on the output.

Figure 16(e) shows that effects of a change in the aerodynamics corresponding to a change in altitudes of 10,000 and 20,000 feet are relatively negligible. For this case the  $D(z)$  was left unchanged at the design value of 30,000 feet altitude. This insensitivity is principally due to the relatively "low-gain" system. Had a more rapid response been required the same conclusions might not have been reached.

It is of interest to compare the transient responses of the sampled-data system derived by this design procedure with those of the continuous system which was designed to obtain desired dominant mode characteristics. The root-locus plot of the continuous system used for comparison is shown in figure 11. Figures 17(a) and (b) show the very close comparison of the transient responses of the two systems. It should be noted that the start of the sampled-data system response was shifted so that the two start at the same point since a delay of 0 to 0.25 second (dependent on the time of the application of the step) can be experienced by the sampled system.

#### CONCLUDING REMARKS

It has been demonstrated that the finite settling time design of a sampled-data system does not produce desirable characteristics when the method is applied to the design of an altitude autopilot. Two undesirable features are the very abrupt response and the extreme sensitivity to parameter changes which are unsatisfactory for aircraft and many other automatic control systems. Increasing the sampling period generally reduces the abruptness of response; however, it does not appear to eliminate the sensitivity problem. The sampling period may also be dictated by other considerations, so that it might not be a variable.

It has been demonstrated that basing the sampled-data system dominant mode on a continuous system design results in a much more practical system with respect to the two previous considerations. It has also been shown that cancellation of aircraft poles by their inverse in the stabilizing digital network is undesirable. This, of course, is not surprising since the same conclusion is true in continuous systems.

The most satisfactory design method tested was one in which no cancellation was allowed and the resulting system has the simplest digital control function possible.

In general, the use of the  $Z$  transform for sampled-data system design appears to be almost as easy to handle as the Laplace transform for continuous system design. One drawback which has been noticed is that root-locus methods are not easily applicable, principally because the  $Z$  transform of the open-loop system has almost the same number of zeros as poles. Since stability requires poles to be confined to the unit circle, a knowledge of the asymptotic behavior for a large  $z$  does not help. In addition a solution for saddle points in the  $z$  plane is almost impossible. This means that one must use the tedious method of trying points to see if they are on the loci without knowing the approximate loci. The other drawback is that for the high order system of this example the sampling period should be chosen initially rather than carried through as an arbitrary constant to be determined in the final design. This is simply because of the complexity of the resultant equations which occur if it is arbitrary. Since there does not appear to be a clear cut means of selecting the sampling period, then one would, in general, have to duplicate several designs at different periods in order to arrive at the most suitable one for the particular application.

Ames Research Center

National Aeronautics and Space Administration  
Moffett Field, Calif., Jan. 14, 1959

## APPENDIX A

## SIMULATION OF THE SAMPLED-DATA SYSTEM

A sampled-data system can be simulated on a conventional analog computer. The only additional device required is a sample-hold circuit. There are many ways of constructing this type device utilizing relays or various electronic circuits. The circuitry used for this study is briefly described in appendix B and is an all electronic device.

Given a sample-hold circuit, how is  $D(z)$  simulated? The method used is described in reference 6; however, the technique is simple and will be described here. The following equations can be developed from figure 18

$$D(z) = \frac{A(z)B(z)}{Z[(1 - e^{-sT})/s]} = A(z)B(z) \quad (A1)$$

$$A(z) = \frac{Z[(1 - e^{-sT})/s]}{1 + Z\{[(1 - e^{-sT})/s]Q(s)\}} = \frac{1}{1 + (1 - z^{-1})Z[Q(s)/s]} \quad (A2)$$

from equation (A2)

$$Z\left[\frac{Q(s)}{s}\right] = \left[\frac{1}{A(z)} - 1\right] \frac{1}{(1 - z^{-1})} \quad (A3)$$

$$B(z) = Z\left[\frac{1 - e^{-sT}}{s} P(s)\right] Z\left(\frac{1 - e^{-sT}}{s}\right) = (1 - z^{-1})Z\left[\frac{P(s)}{s}\right] \quad (A4)$$

from equation (A4)

$$Z\left[\frac{P(s)}{s}\right] = \frac{B(z)}{(1 - z^{-1})} \quad (A5)$$

A given  $D(z)$  is to be simulated by the circuit of figure 18. The problem is then to split  $D(z)$  into  $A(z)$  and  $B(z)$  so that  $Q(s)$  and  $P(s)$  can be determined from equations (3) and (5), respectively.

$$D(z) = K \left[ \frac{(1 - a_1 z^{-1})(1 - a_2 z^{-2})(. . .)}{(1 - b_1 z^{-1})(1 - b_2 z^{-2})(. . .)} \right] \quad (A6)$$

Now the following conditions insure that  $Q(s)$  and  $P(s)$  will be physically realizable stable networks:

- (1) All unstable poles of  $D(z)$  shall be contained in  $A(z)$ .
- (2) Unstable zeros of  $D(z)$  shall be contained in  $B(z)$ .
- (3) All gain,  $K$ , shall be contained in  $B(z)$ .
- (4) Stable poles and zeros shall be distributed between  $A(z)$  and  $B(z)$  to allow  $\{[1/A(z)] - 1\}[1/(1 - z^{-1})]$  and  $B(z)/(1 - z^{-1})$  to be expanded in partial fractions of type  $\beta/(1 - \alpha z^{-1})$ .

Reference 6 gives a number of examples and explains in detail how all physically realizable  $D(z)$ 's may be constructed from R-C networks. Since the desire here is to simulate, one may allow complex poles. Conditions 1, 2, and 4 assure  $Q(s)$  and  $P(s)$  will be stable networks. These conditions certainly do not appear to be necessary for simulation purposes, a fact which essentially means that all of  $D(z)$  except the gain,  $K$ , can be assigned to  $A(z)$ . In figure 18, this can be seen to be a desirable way of handling the simulation since only one sample-and-hold circuit is required. For the example, Case I,  $K_q = 0$ ,  $T = 0.25$

$$D(z) = -0.1579 \left[ \frac{(1 - 0.7146z^{-1})(1 - 1.356z^{-1} + 0.7182z^{-2})}{1 + 2.215z^{-1} + 1.276z^{-2} - 0.1211z^{-3}} \right] \quad (A7)$$

$$B(z) = -0.1579 = \text{Gain of } D(z) \quad (A8)$$

$$A(z) = \frac{(1 - 0.7146z^{-1})(1 - 1.356z^{-1} + 0.7182z^{-2})}{1 + 2.215z^{-1} + 1.276z^{-2} - 0.1211z^{-3}} \quad (A9)$$

$$Z \left[ \frac{Q(s)}{s} \right] = \left[ \frac{1}{A(z)} - 1 \right] \frac{1}{(1 - z^{-1})} = \frac{4.285z^{-1} - 0.4111z^{-2} + 0.3941z^{-3}}{(1 - z^{-1})(1 - 0.7146z^{-1})(1 - 1.356z^{-1} + 0.7182z^{-2})} \quad (A10)$$

This expression is expanded in partial fraction expansion

$$Z \left[ \frac{Q(s)}{s} \right] = z \left( \frac{41.26}{z - 1} - \frac{30.83}{z - 0.7146} - \frac{10.40z + 0.8049}{z^2 - 1.356z + 0.7182} \right) \quad (A11)$$

The terms are individually converted to the  $s$  plane and recombined to give

$$Q(s) = \frac{8.655s^2 + 84.38s + 392.0}{(s + 1.345)(s^2 + 1.324s + 7.065)} \quad (A12)$$



It should be noted that  $Q(s)$  has a lower order numerator than denominator. This is absolutely essential because of the circuit used and is a consequence of imposing condition 3.

A block diagram of the system to be simulated is shown in figure 19. Figure 20 shows the corresponding analog-computer diagram used for simulating the system.

## APPENDIX B

## AN ELECTRONIC SAMPLE-HOLD CIRCUIT

As was previously mentioned, there are a number of methods of constructing a sample-hold circuit. The circuitry utilized for this example is all electronic and for some applications may offer advantages over circuitry utilizing relays.

Figure 21 is a block diagram of the circuit used. The diode bridge acts as a gate circuit connecting the output of amplifier 1 to the input of integrator 2. When a positive pulse is applied to A and a negative pulse to B, the diode bridge connects the output of amplifier 1 to the input of integrator 2. When zero signal is applied to A and B the input to integrator 2 is open; thus, the diode bridge acts as a sampler of amplifier 1. The function of integrator 2 is to hold the output between the sampling instants. Integrator 2 acts as an open-loop integrator during the time in which the pulse is off and, as a consequence, its output (other than for drift) stays at the value it was at the previous sampling instant. If an initial step of  $E_i$  is assumed the circuit can be seen to work in the following manner. Initially amplifier 1 builds up to the step value; as soon as the pulse arrives at A and B, the integrator output starts to change value. If the time constant of amplifier 1, 2, 3 combination is about  $1/5$  of the sampling time, then the output  $E_o$  will be equal to  $+E_i$  at the end of the sample time. Then  $E_o$  is held at  $E_i$  by the integrator condenser. Drifts are corrected each sampling instant. For this problem it was desired that the sample time should not be longer than 1 percent of the sample period, T, and that the problem be run on a real time basis on the analog computer. This means that for a 4 cps sampling frequency the sample time  $\approx 1/100 \times 0.25$  second = 2.5 milliseconds and the time constant of amplifier 1, 2, 3 combination should be approximately 0.5 millisecond.

As can be seen from the above number, relatively fast amplifiers must be available for use in this sample-hold circuit. This is one difficulty which would probably, in general, make this circuit difficult to fabricate from conventional analog computer amplifiers. For use in this problem specially designed high-performance chopper-stabilized amplifiers were available. No particular problems were noticed in the use of this circuit other than the need for an occasional adjustment to compensate for drift in the integrator.

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1. Ragazzini, J. R., and Franklin, G. F.: Sampled-Data Control Systems. McGraw Hill, 1958.
2. Barker, R. H.: The Theory of Pulse-Monitored Servos and Their Use for Prediction. Rep. 1046, Signals Research and Development Establishment, Christ Church, England, Nov. 1950.
3. Bertram, John E.: Factors in the Design of Digital Controllers for Sampled-Data Feedback Systems. Communications and Electronics, vol. 75, no. 25, July 1956, pp. 151-159. (Also available as Columbia Univ. TR T-8/B, Aug. 15, 1955, and OSR TN 55-360)
4. Truxal, John G.: Automatic Feedback Control System Synthesis. McGraw Hill, 1955.
5. Gardner, Murray F., and Barnes, John L.: Transients in Linear Systems. John Wiley and Sons, Inc., 1942.
6. Sklansky, Jack: Network Compensation of Error-Sampled Feedback Control Systems. TR T-7/B, Electronics Res. Lab., Dept. Elect. Eng., Columbia Univ., N. Y., April 1, 1955. (Also available as OSR TN 55-162)



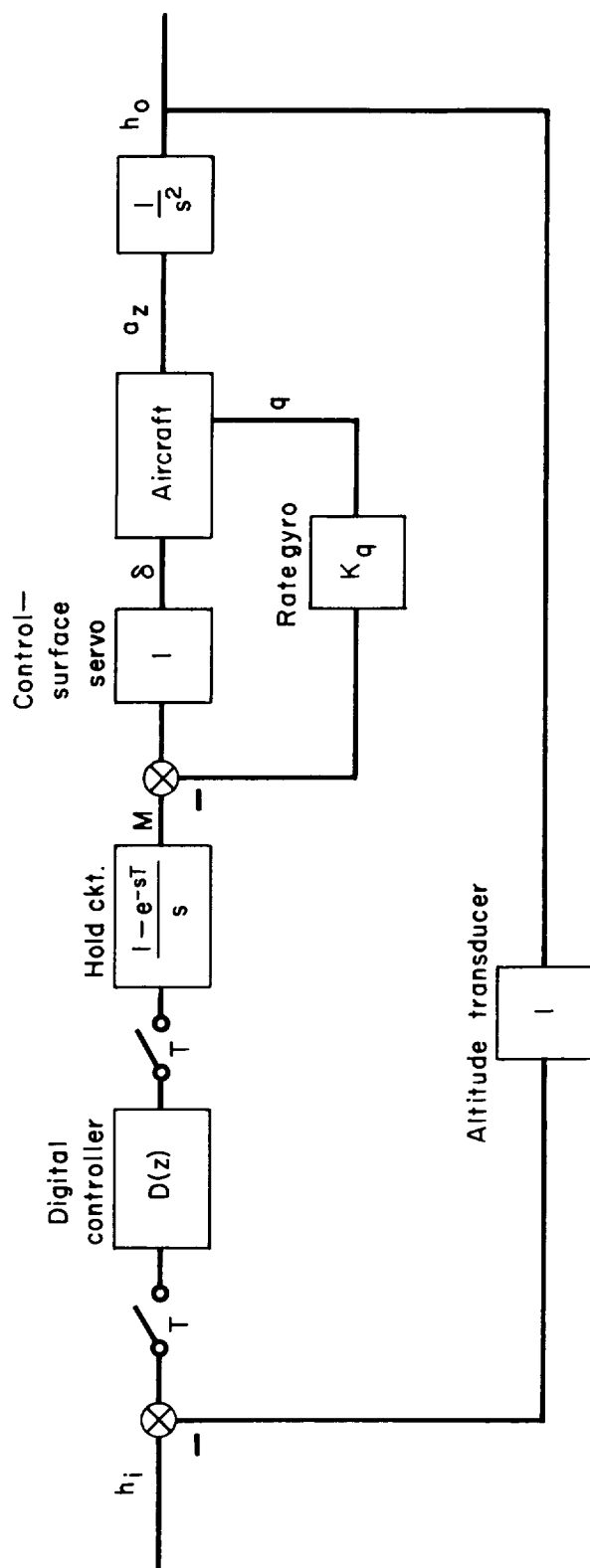
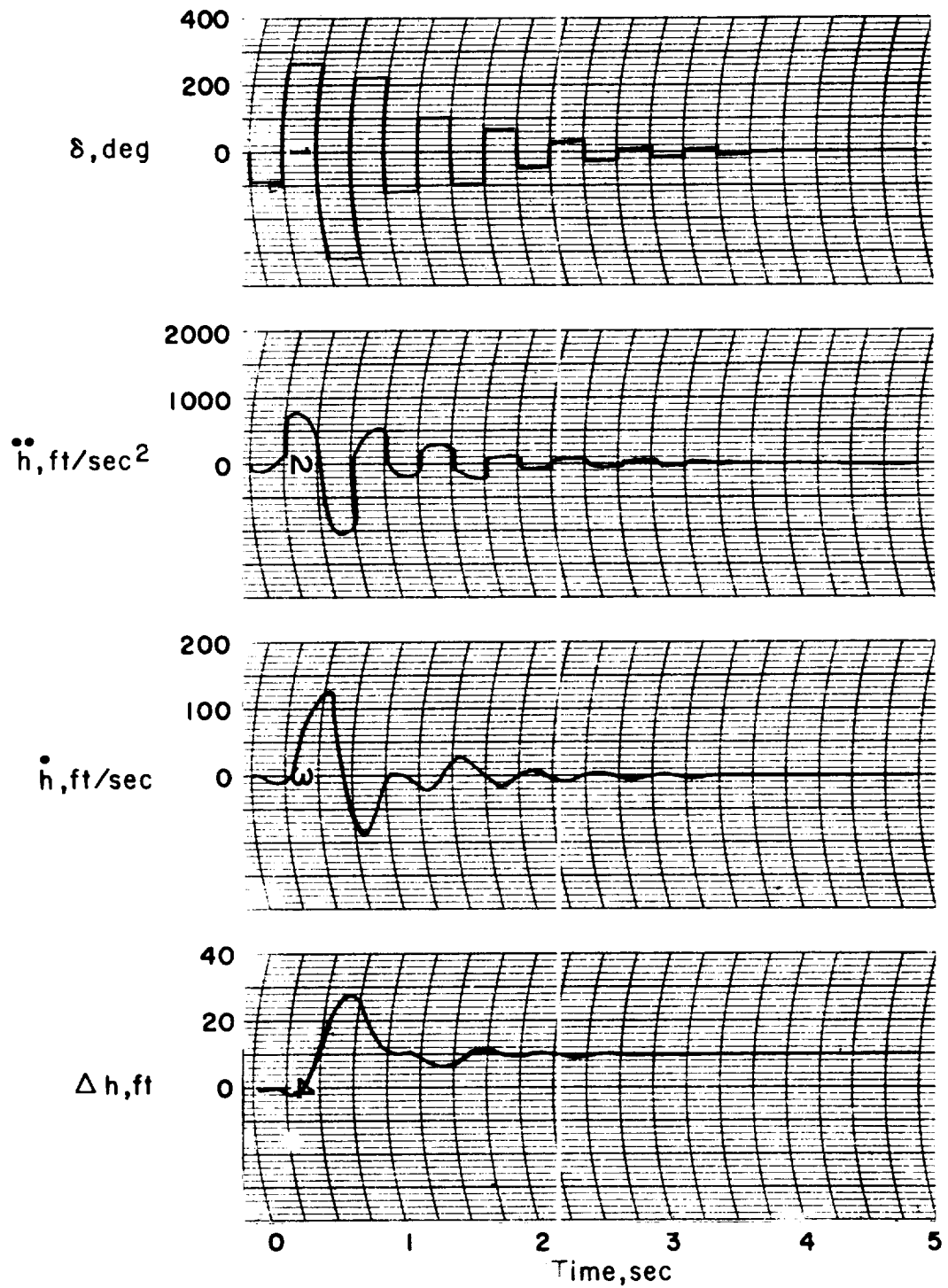
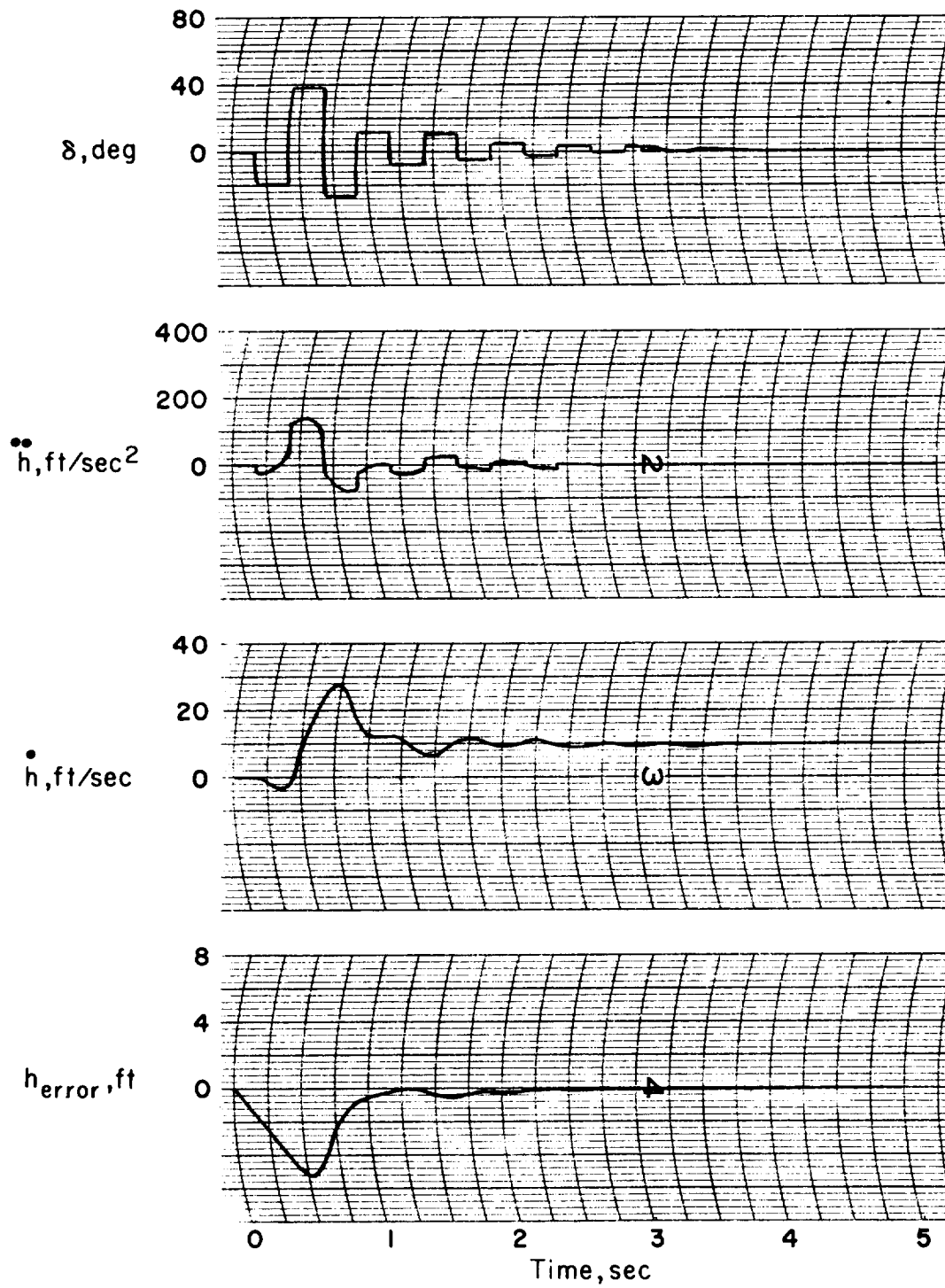


Figure 1.- Block diagram of an altitude autopilot.



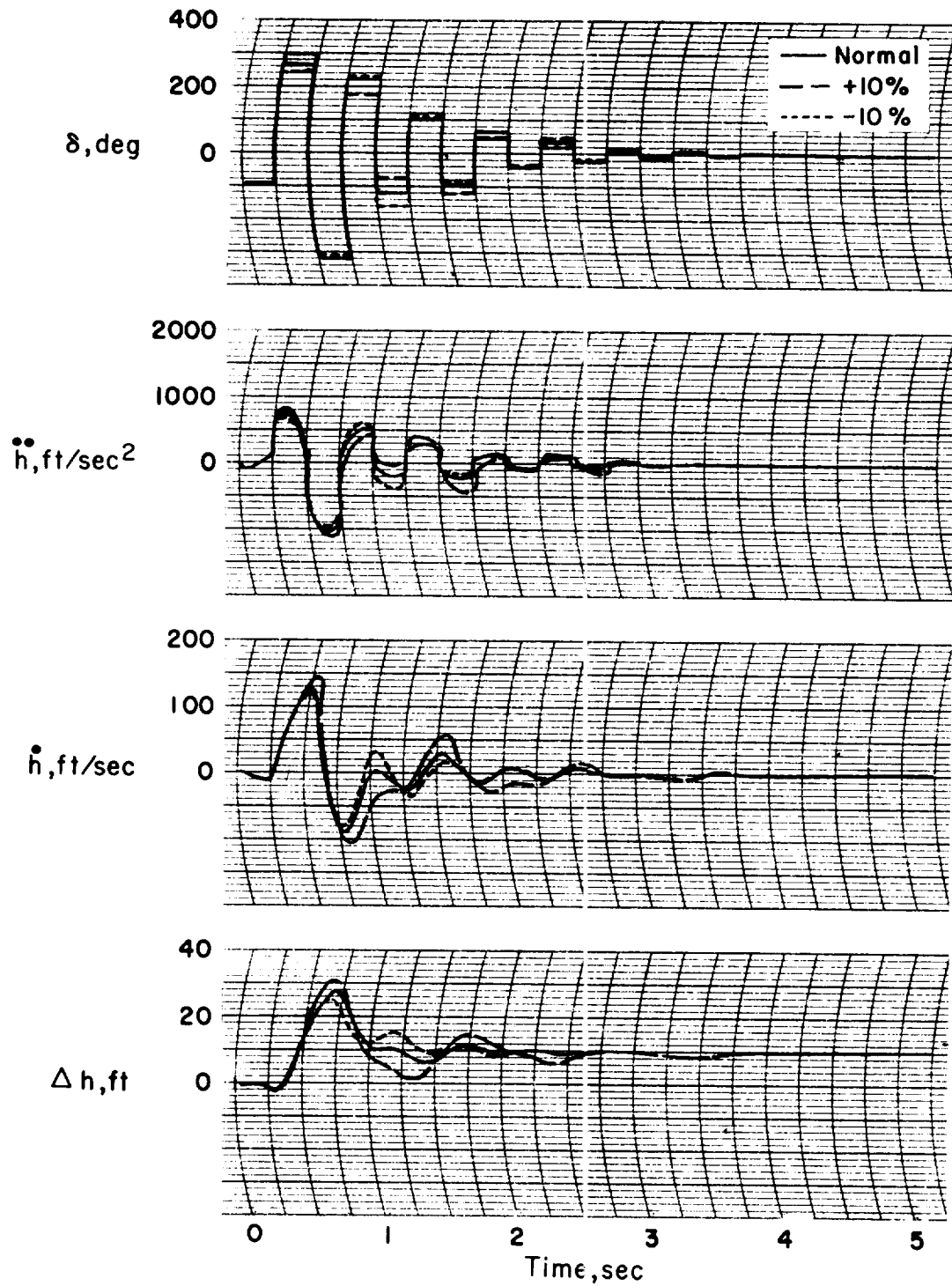
(a) 10-foot step input.

Figure 2.- Transient responses for a finite settling time design;  
Case I,  $T = 0.25$ .



(b) 10-foot per second ramp input.

Figure 2.- Continued.



(c) Effects of  $\pm 10$ -percent gain variation; 10-foot step input.

Figure 2.- Concluded.



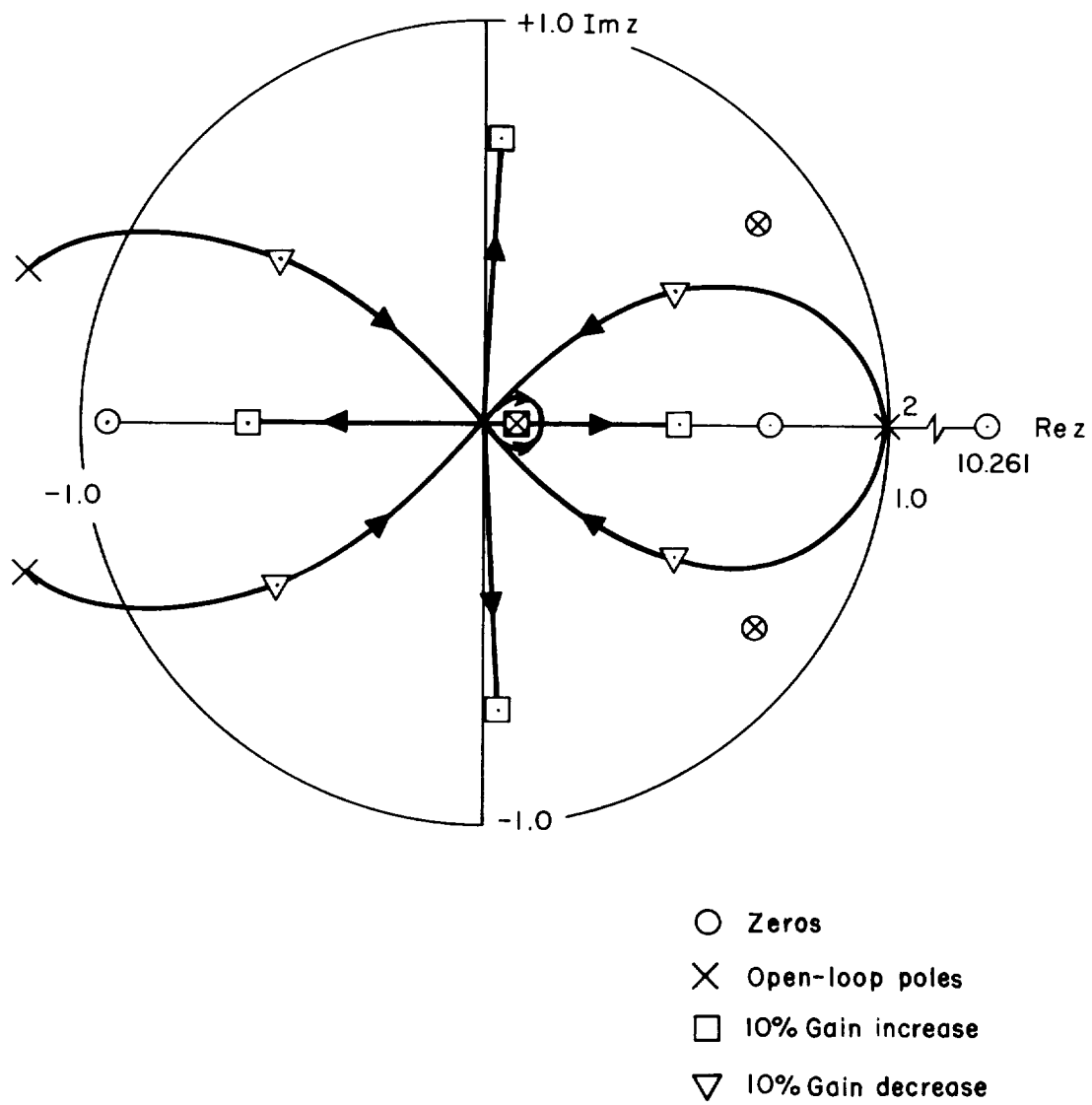
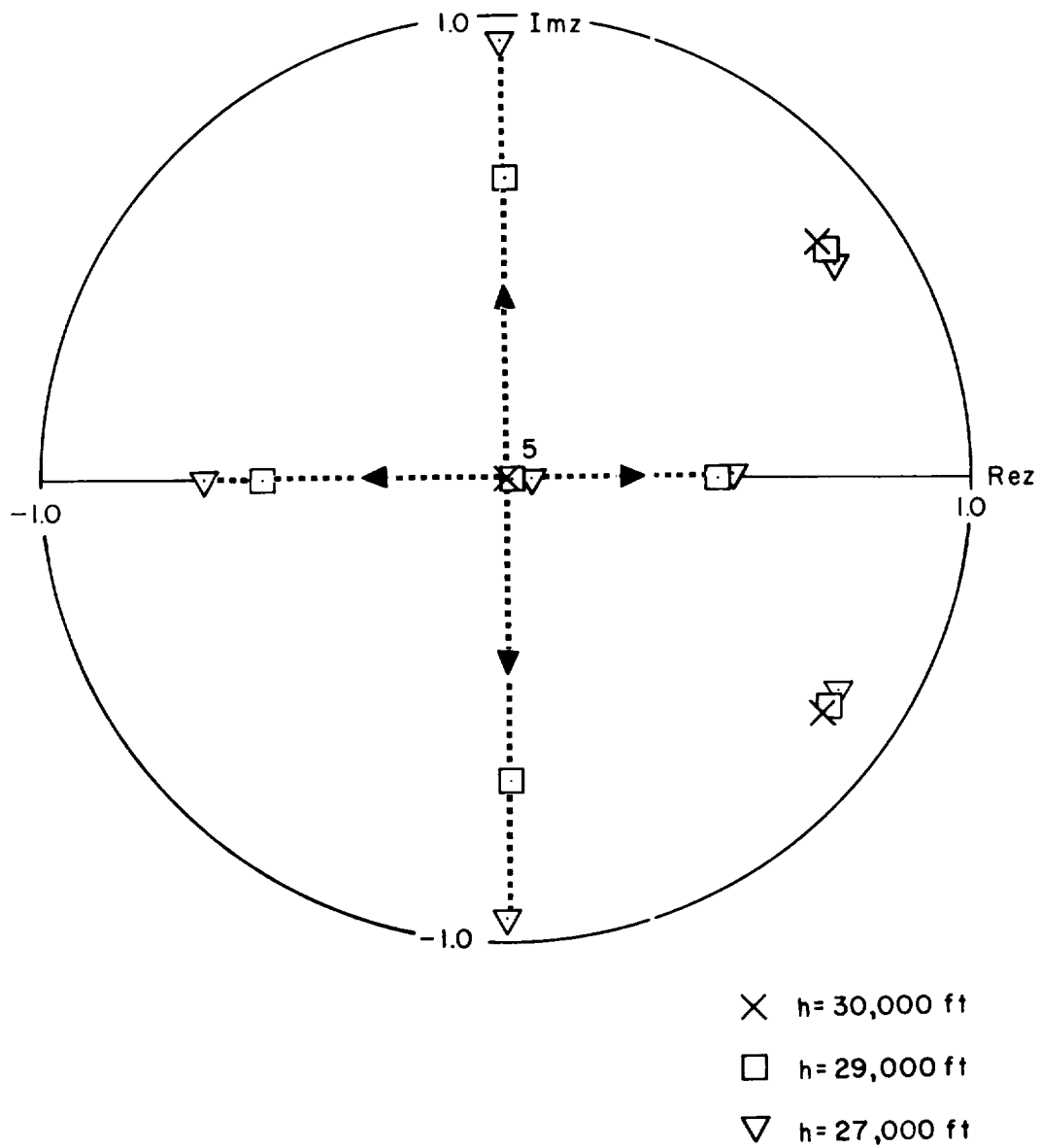
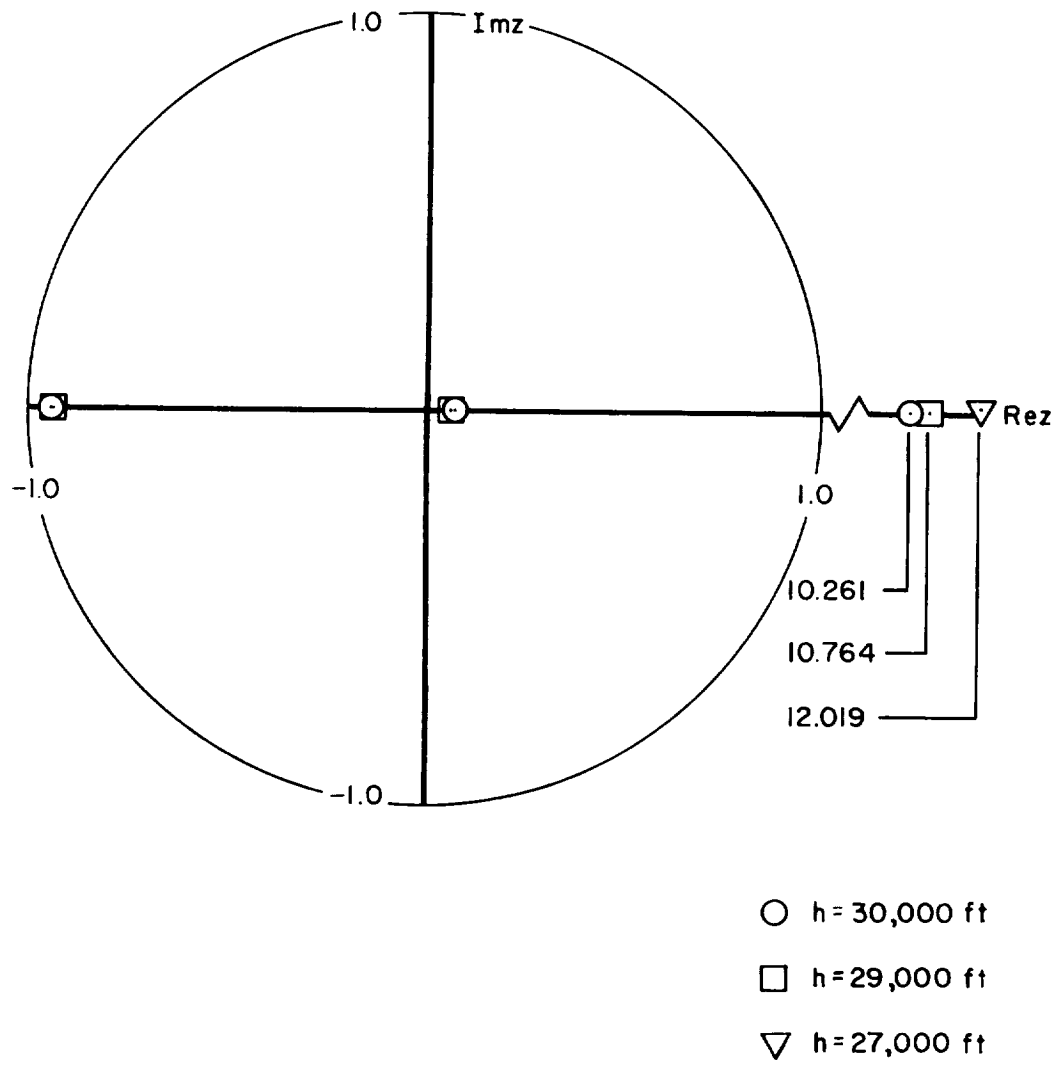


Figure 3.- Root locus for a finite settling time design showing effects of open-loop gain; Case I,  $T = 0.25$ .



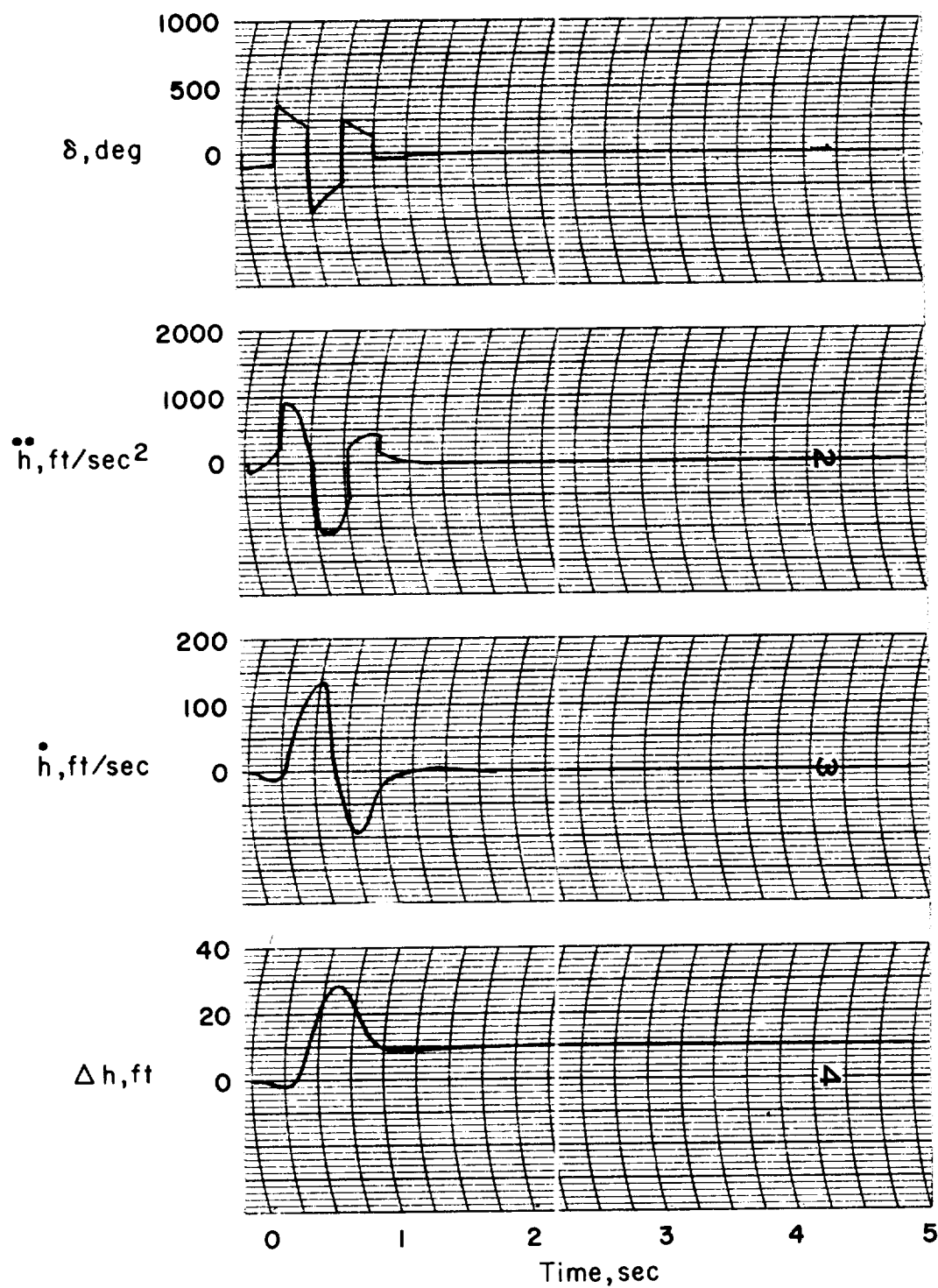
(a) Pole location.

Figure 4.- Effect of altitude on closed loop pole and zero location;  
Case I,  $T = 0.25$ , finite settling time design.



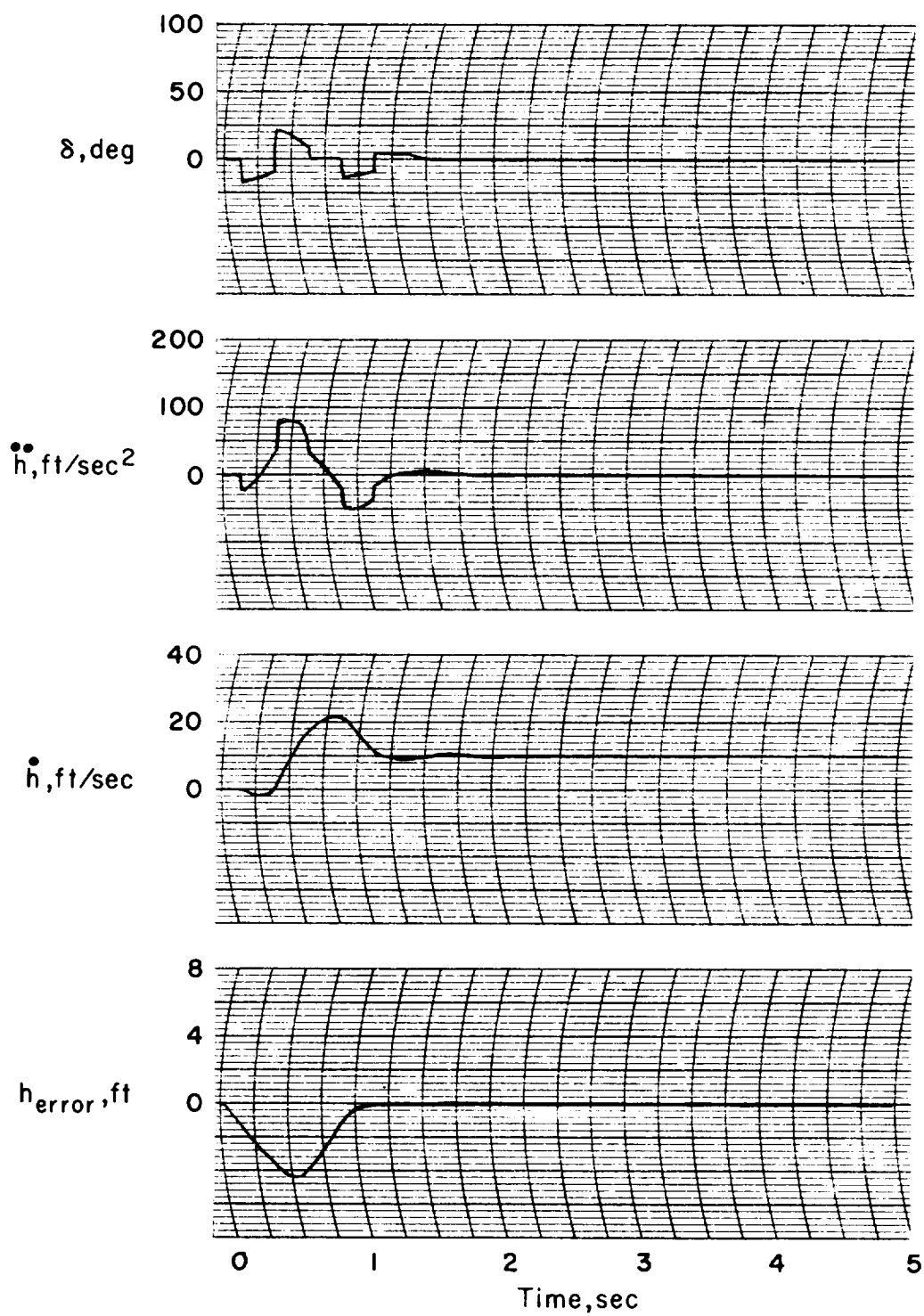
(b) Zero location.

Figure 4.- Concluded.



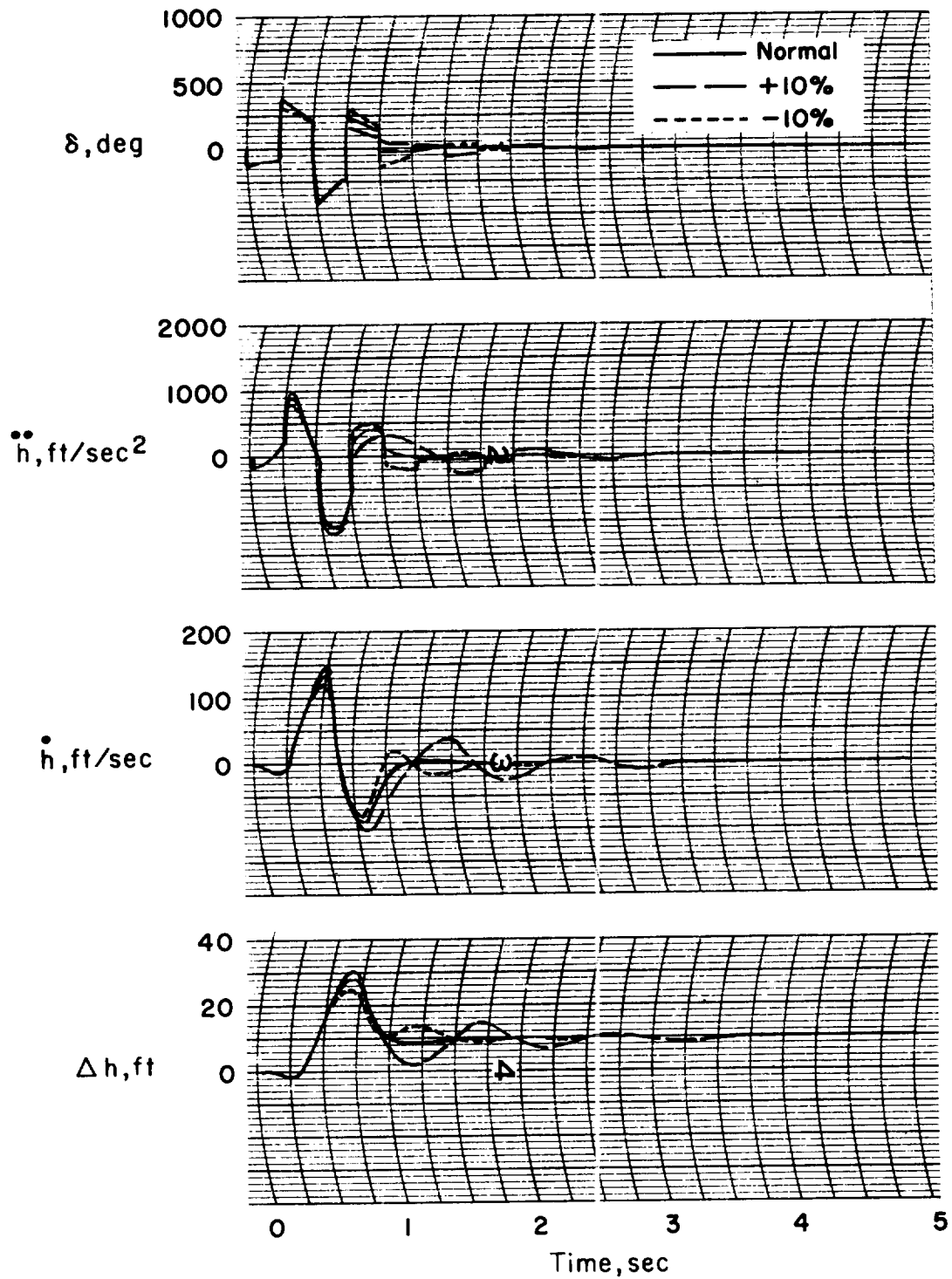
(a) 10-foot step input.

Figure 5.- Transient responses for a finite settling time design;  
Case II,  $T = 0.25$ .



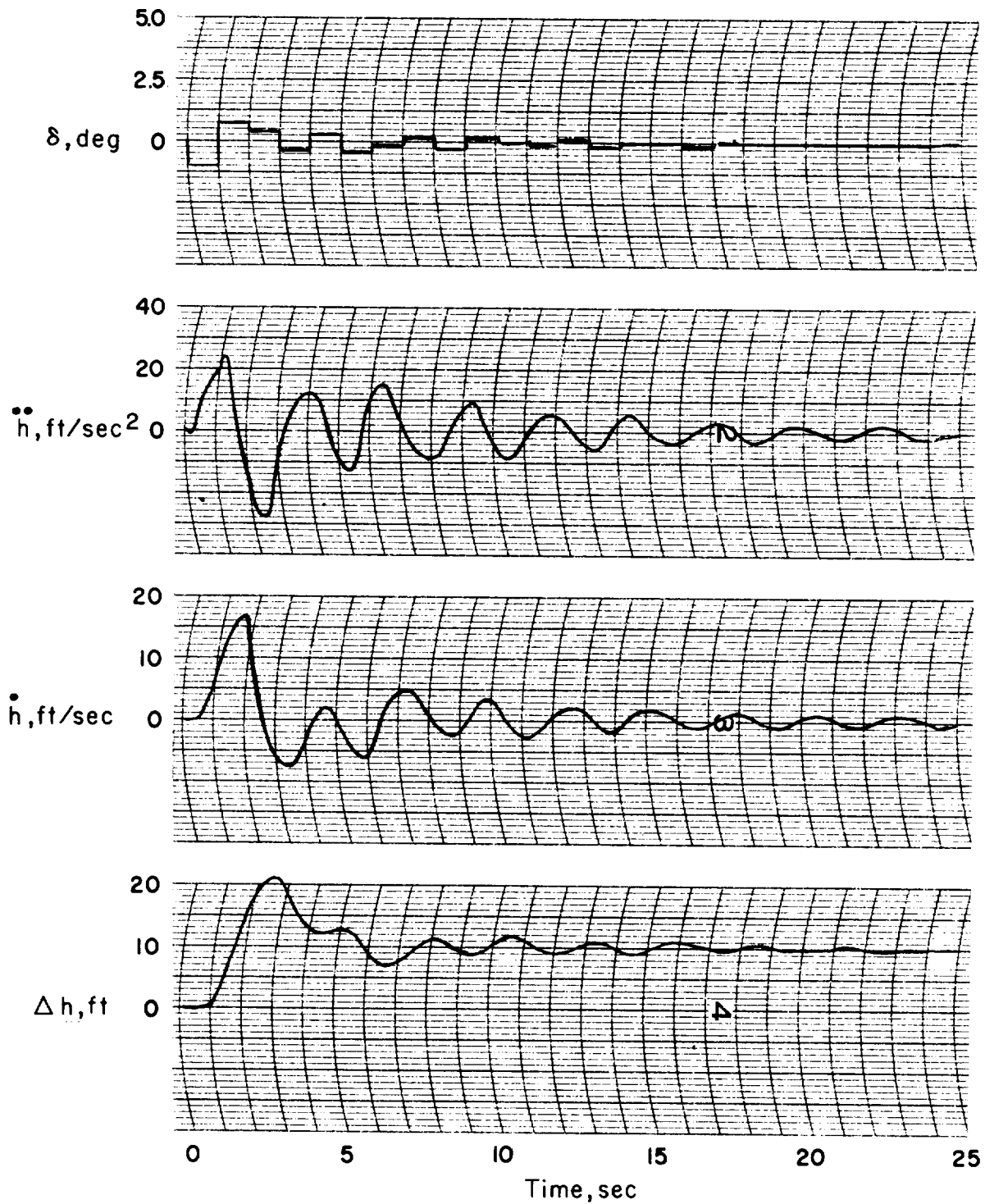
(b) 10-foot per second ramp input.

Figure 5.- Continued.



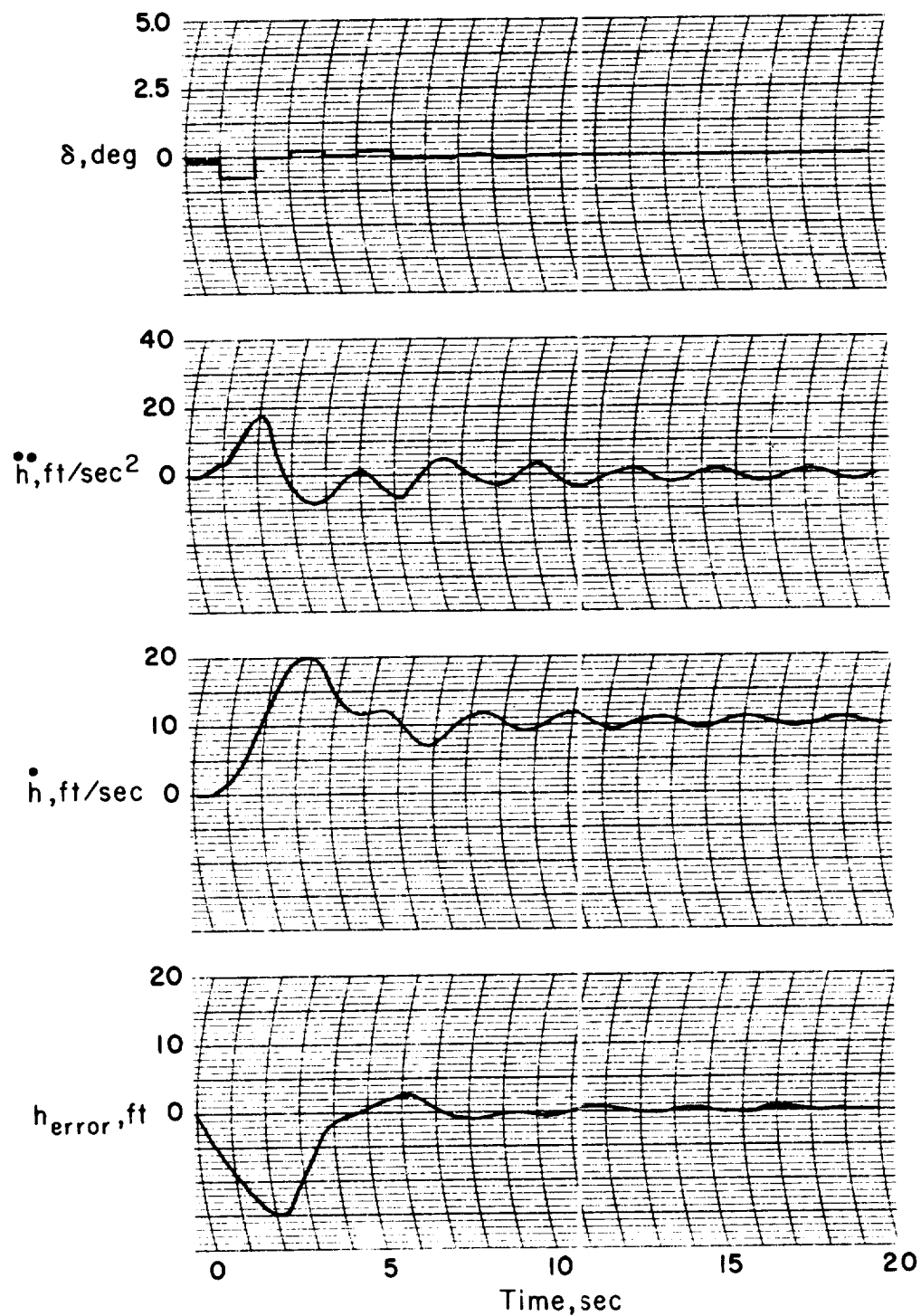
(c) Effects of  $\pm 10$ -percent gain changes; 10-foot step input.

Figure 5.- Concluded.



(a) 10-foot step input.

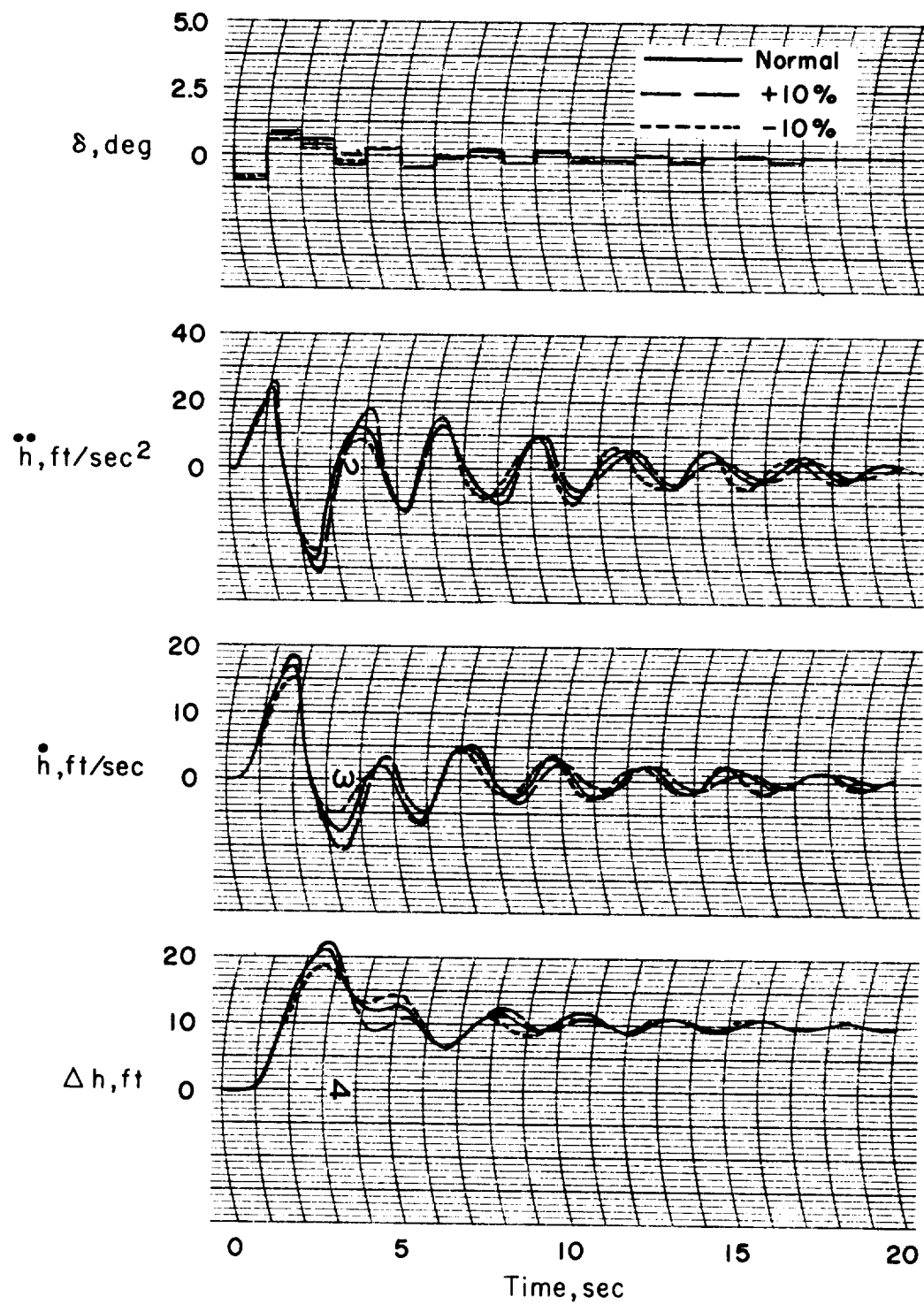
Figure 6.- Transient responses for a finite settling time design;  
Case I,  $T = 1.0$ .



(b) 10-foot per second ramp input.

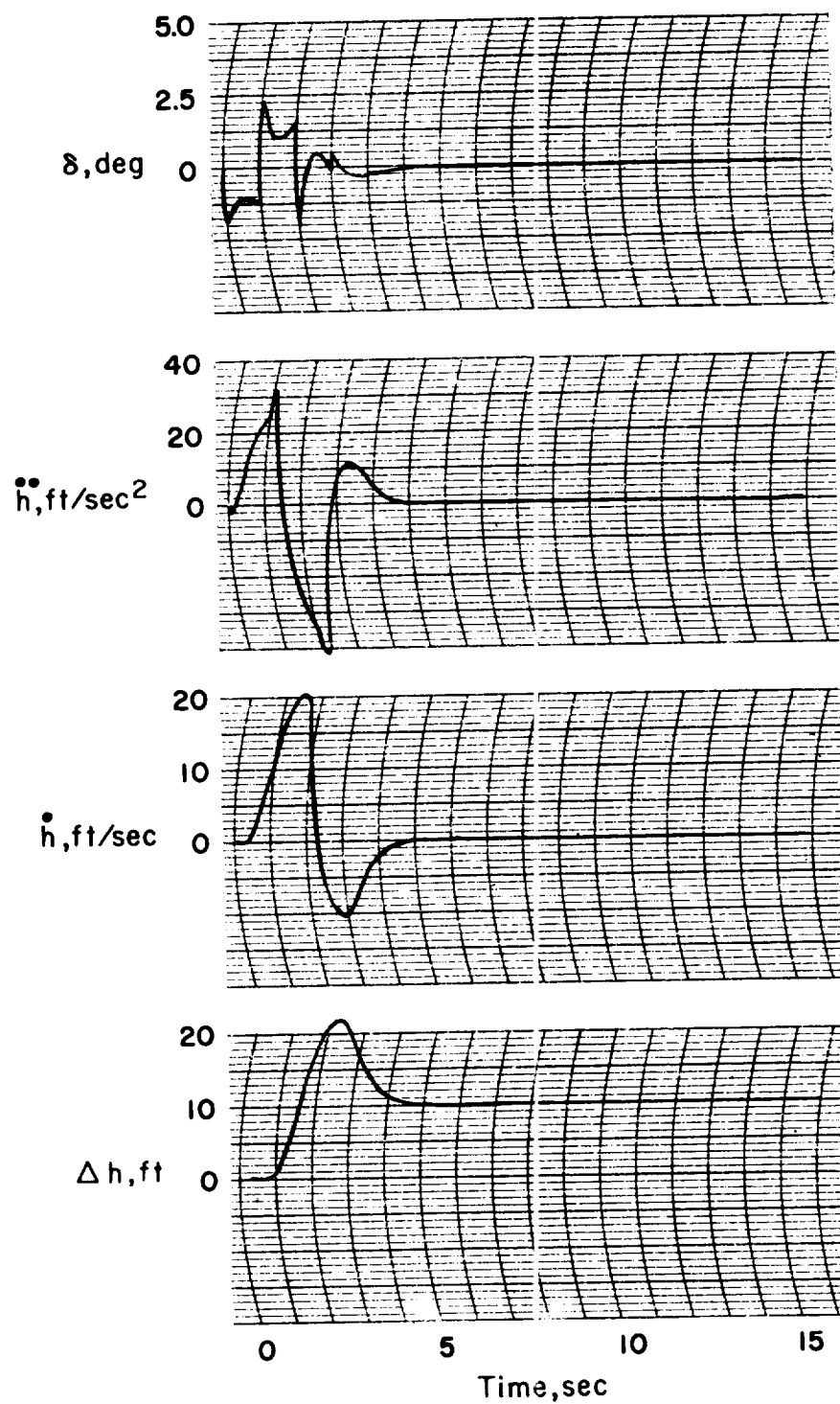
Figure 6.- Continued.





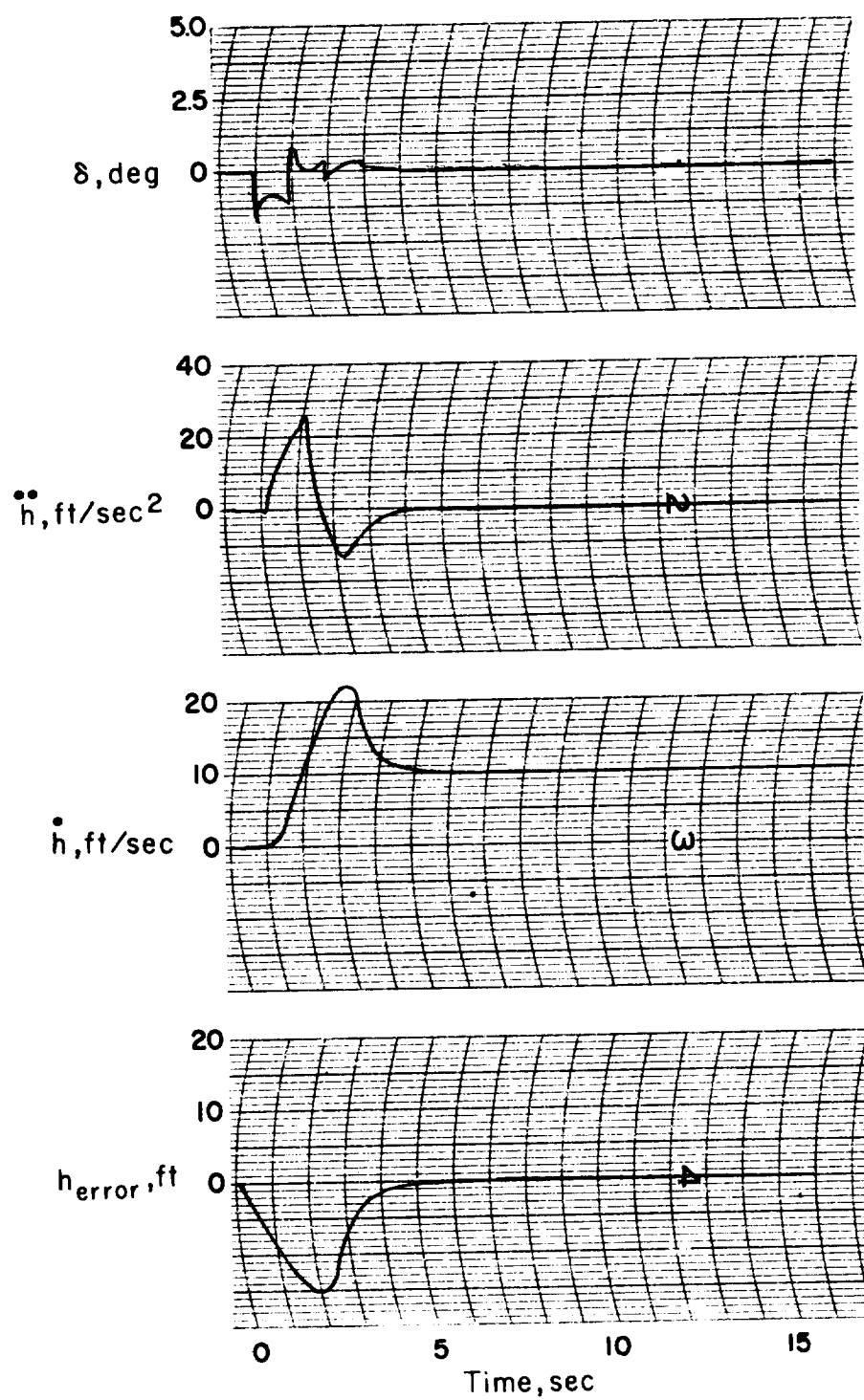
(c) Effects of  $\pm 10$ -percent gain change; 10-foot step input.

Figure 6.- Concluded.



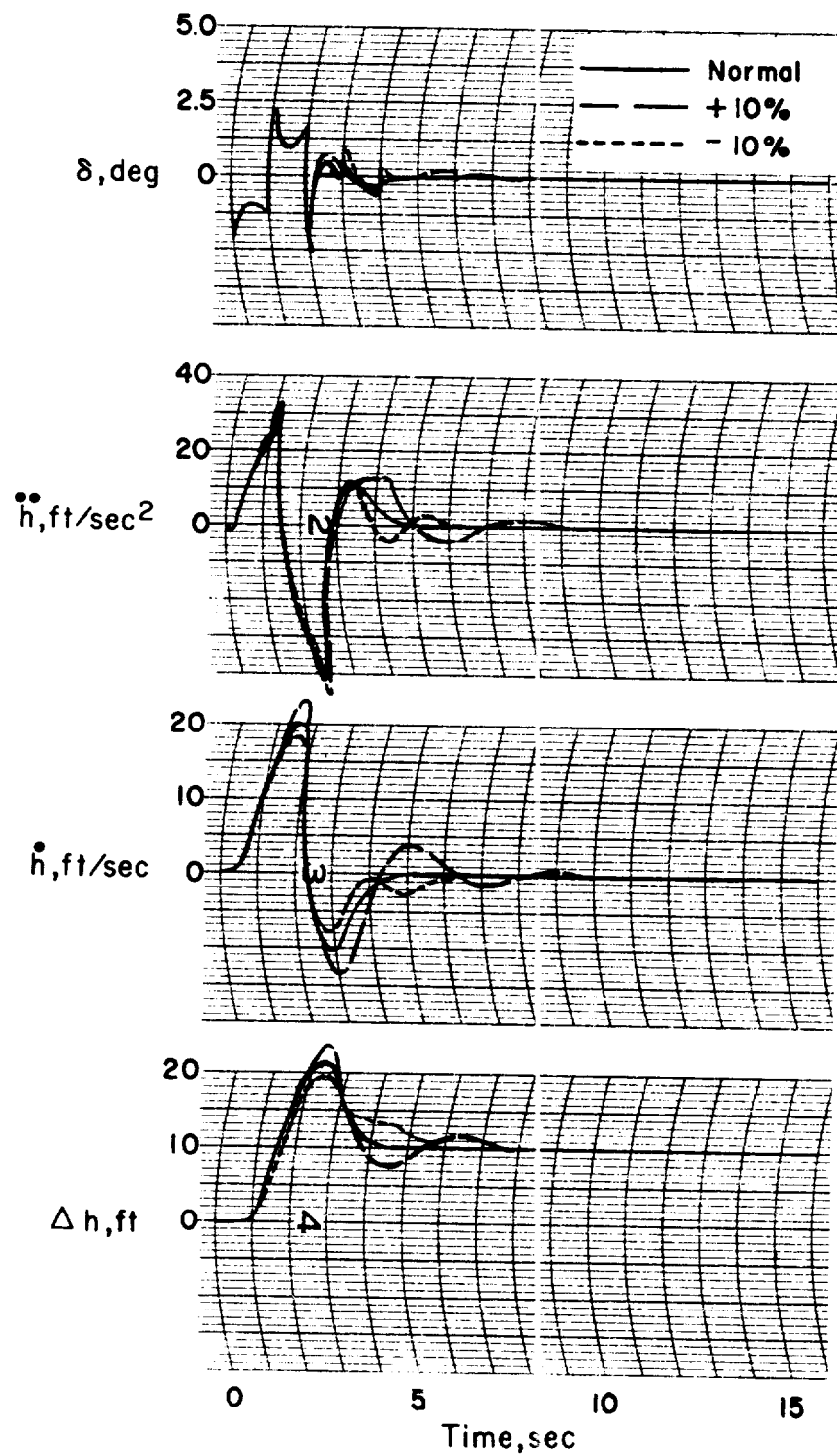
(a) 10-foot step input.

Figure 7.- Transient responses for a finite settling time design;  
Case II,  $T = 1.0$ .



(b) 10-foot per second ramp input.

Figure 7.- Continued.



(c) Effects  $\pm 10$ -percent gain changes; 10-foot step input.

Figure 7.- Concluded.

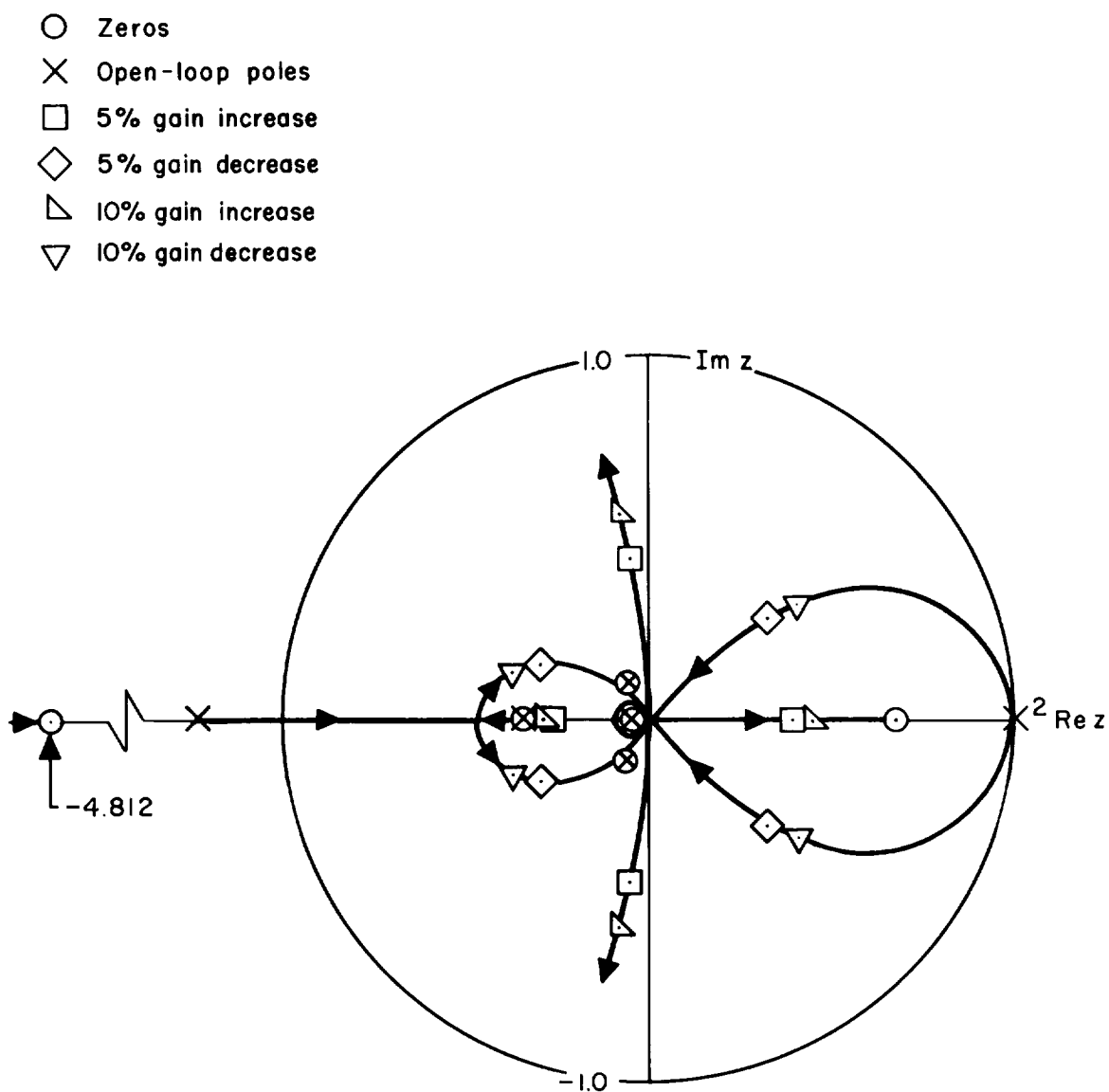
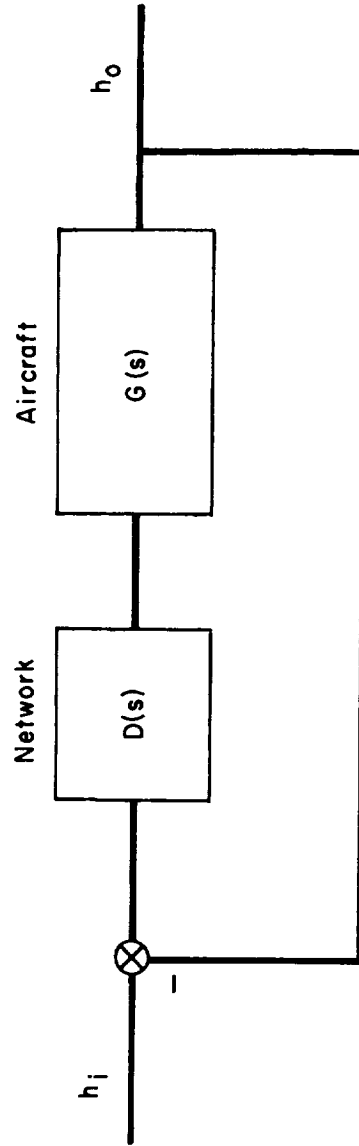


Figure 8.- Root locus for a finite settling time design showing effects of open-loop gain; Case II,  $T = 1.0$ .



$$\text{Case I } G(s) = \frac{83.75(s^2 + .652s - 82.66)}{s^2(s^2 + 1.324s + 7.065)} = \frac{83.75(s + 9.424)(s - 8.772)}{s^2(s + .662 + 2.574i)(s + .662 - 2.574i)}$$

$$\text{Case II } G(s) = \frac{83.75(s^2 + .652s - 82.66)}{s^2(s^2 + 4.206s + 8.858)} = \frac{83.75(s + 9.424)(s - 8.772)}{s^2(s + 2.103 + 2.106i)(s + 2.103 - 2.106i)}$$

Figure 9.- Block diagram of a continuous altitude-command autopilot.

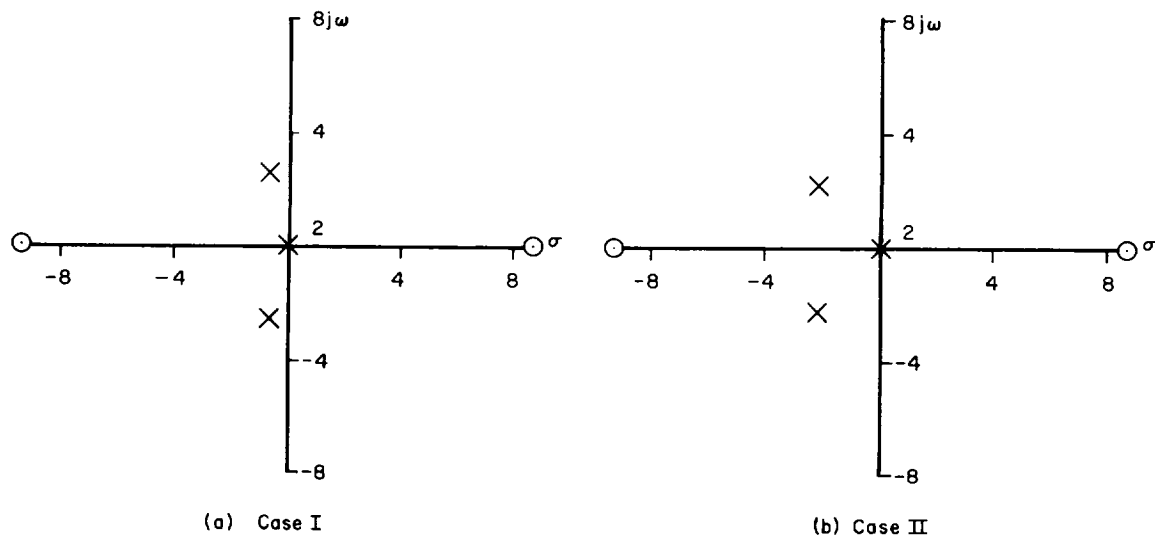


Figure 10.- Location of poles and zeros of  $G(s)$ .

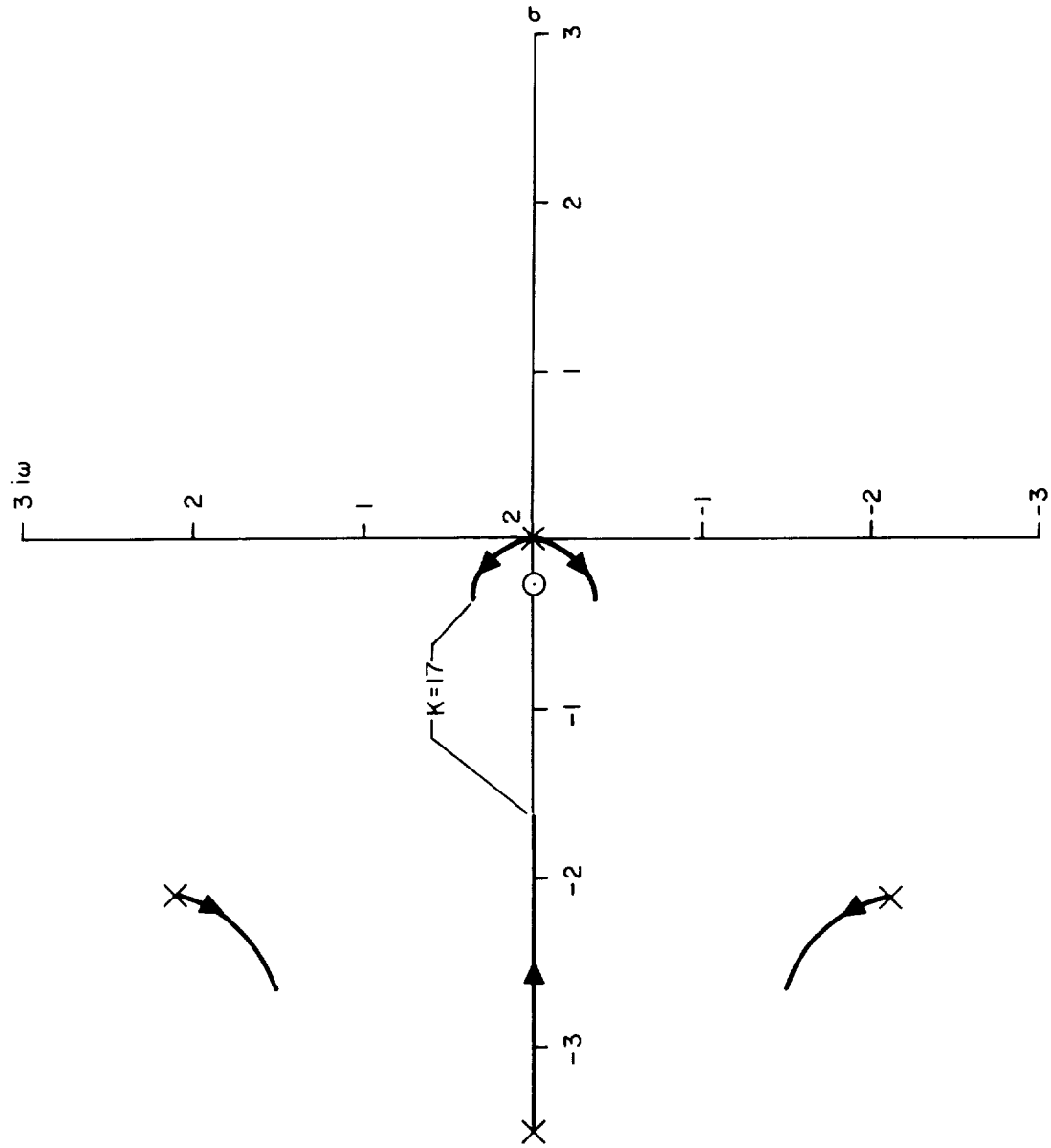


Figure 11.- Root locus for a continuous system; Case II.



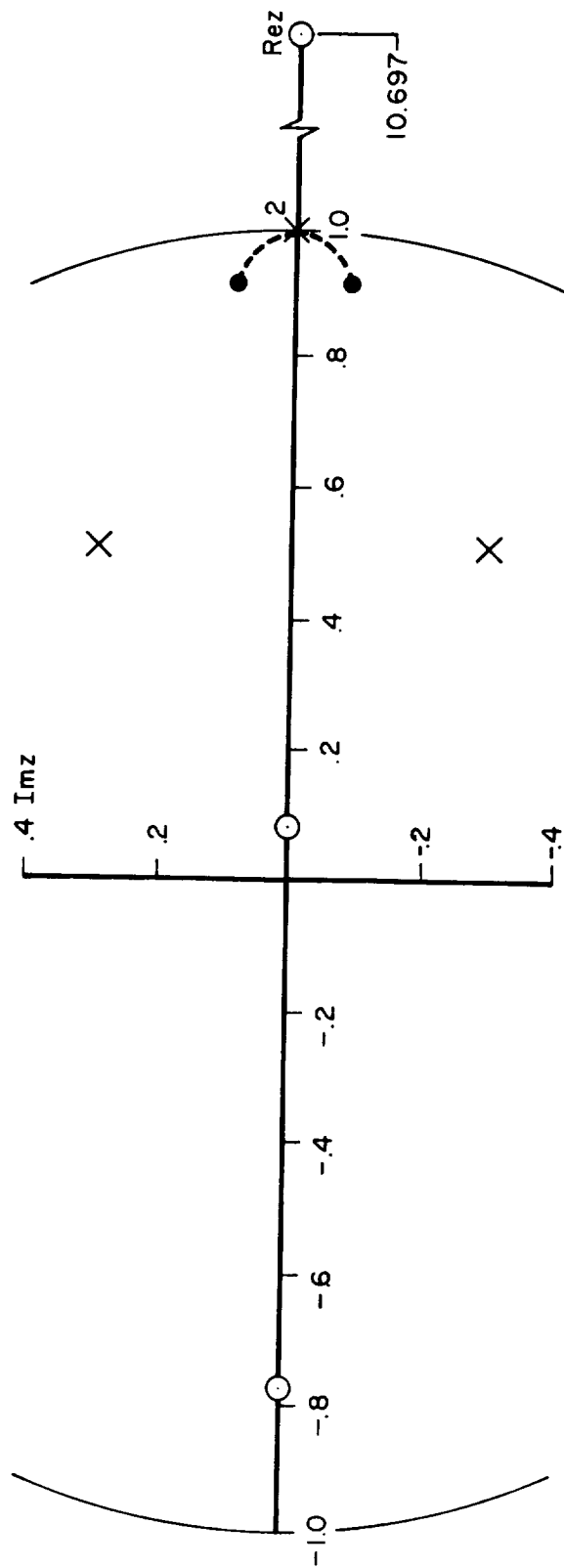
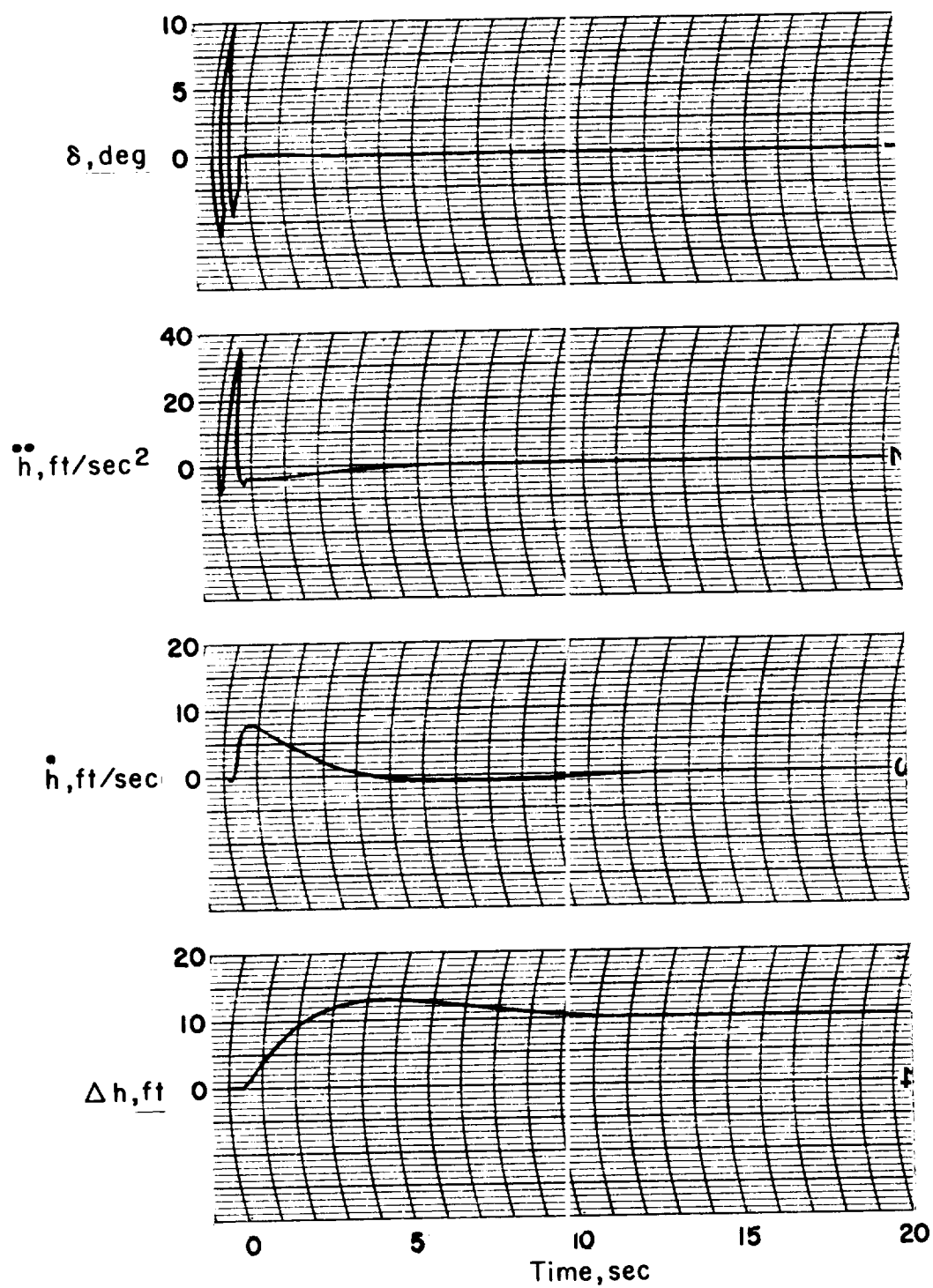
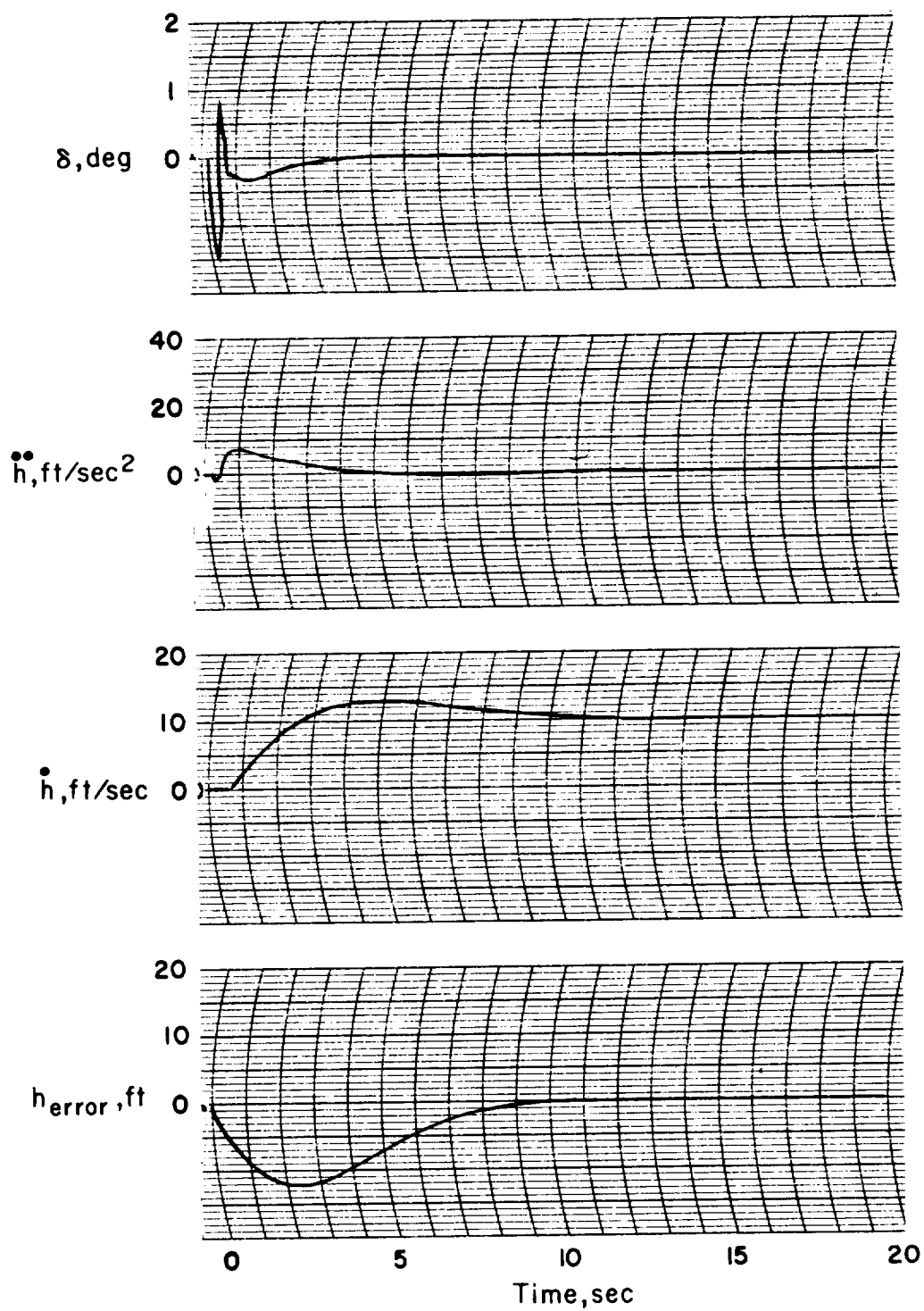


Figure 12.- Location of poles and zeros of  $G(z)$ ; Case II,  $T = 0.25$ .



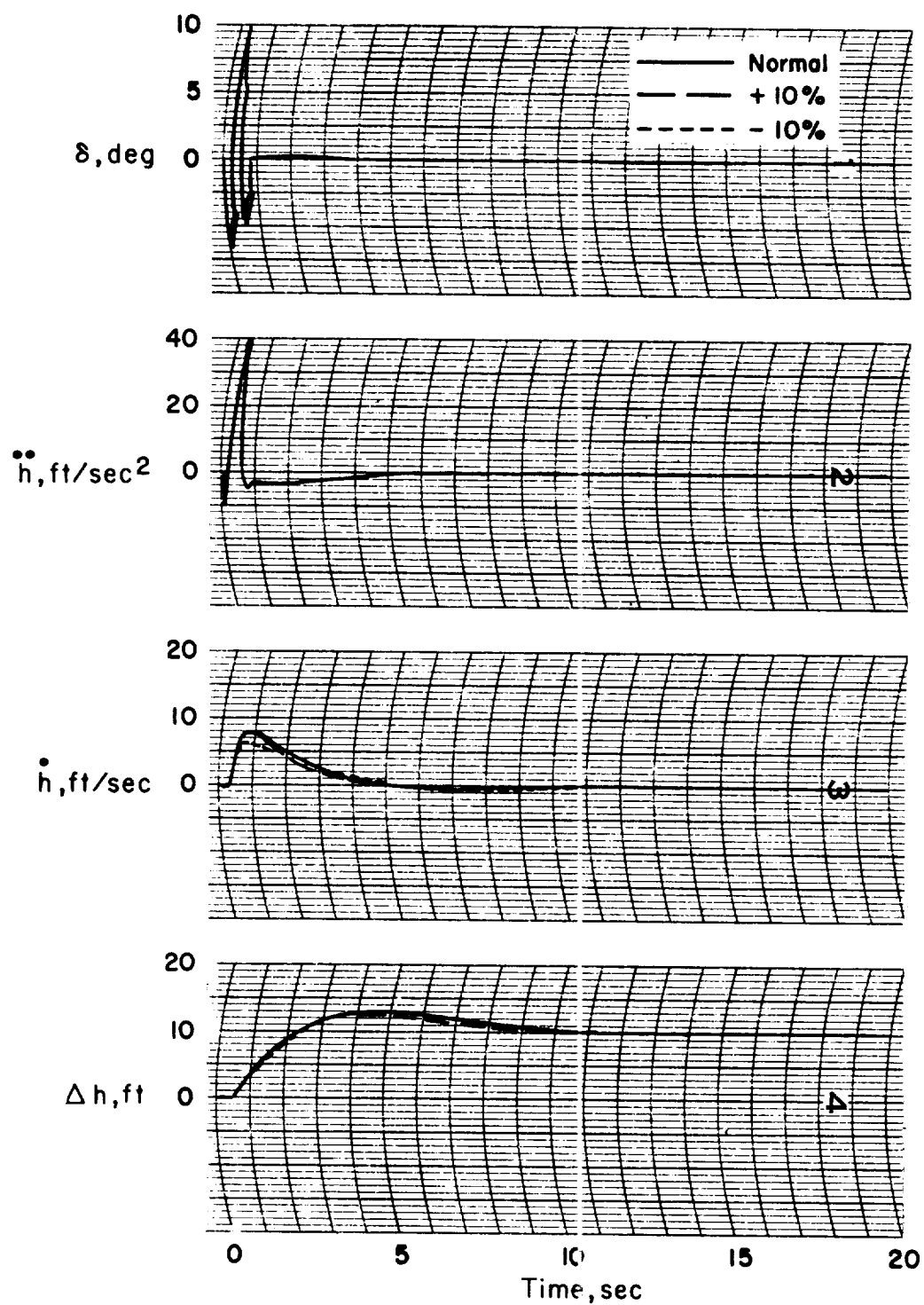
(a) 10-foot step input.

Figure 13.- Transient responses for a modified system, cancellation permitted; Case II,  $\Gamma = 0.25$ .



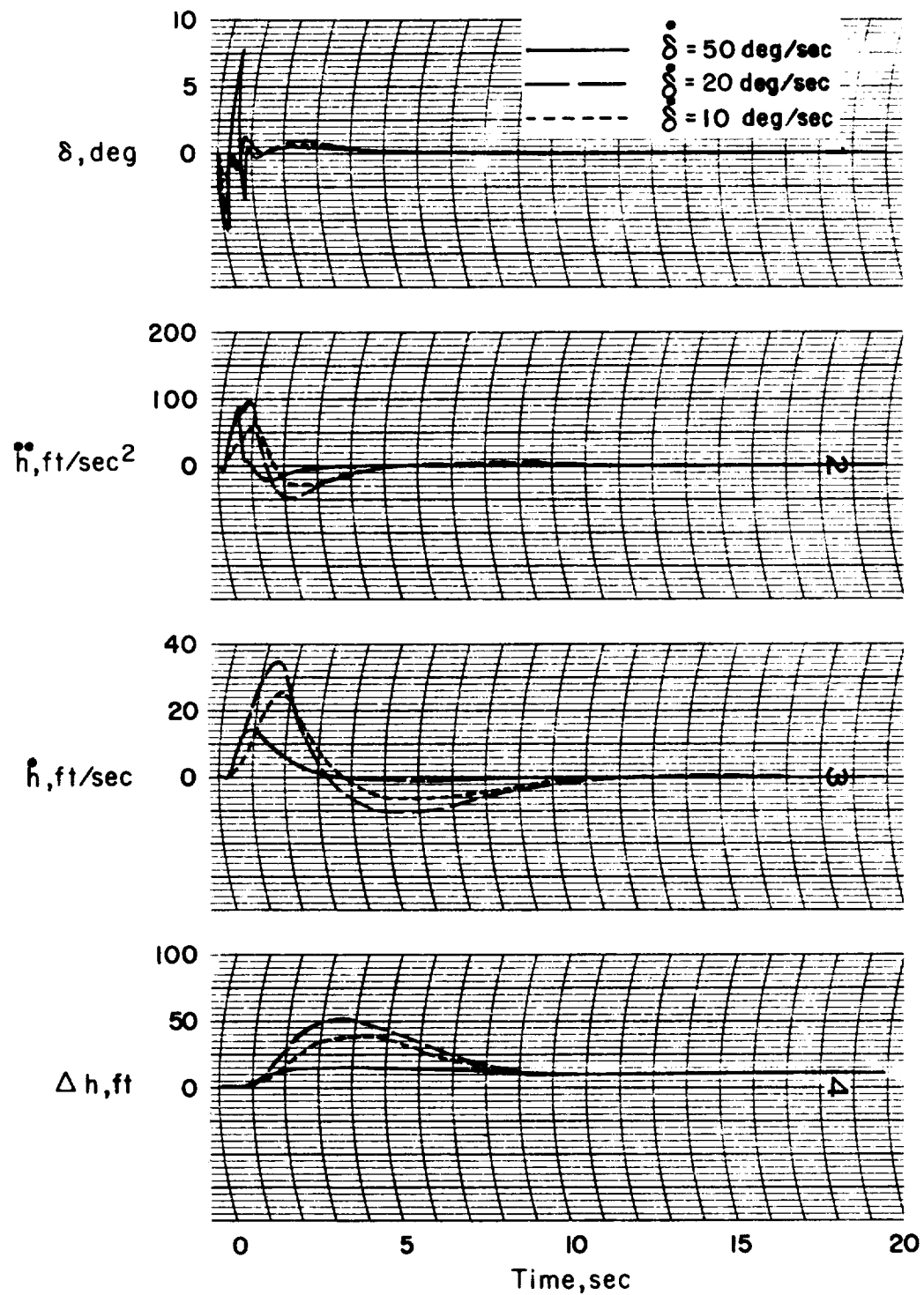
(b) 10-foot per second ramp input.

Figure 13.- Continued.



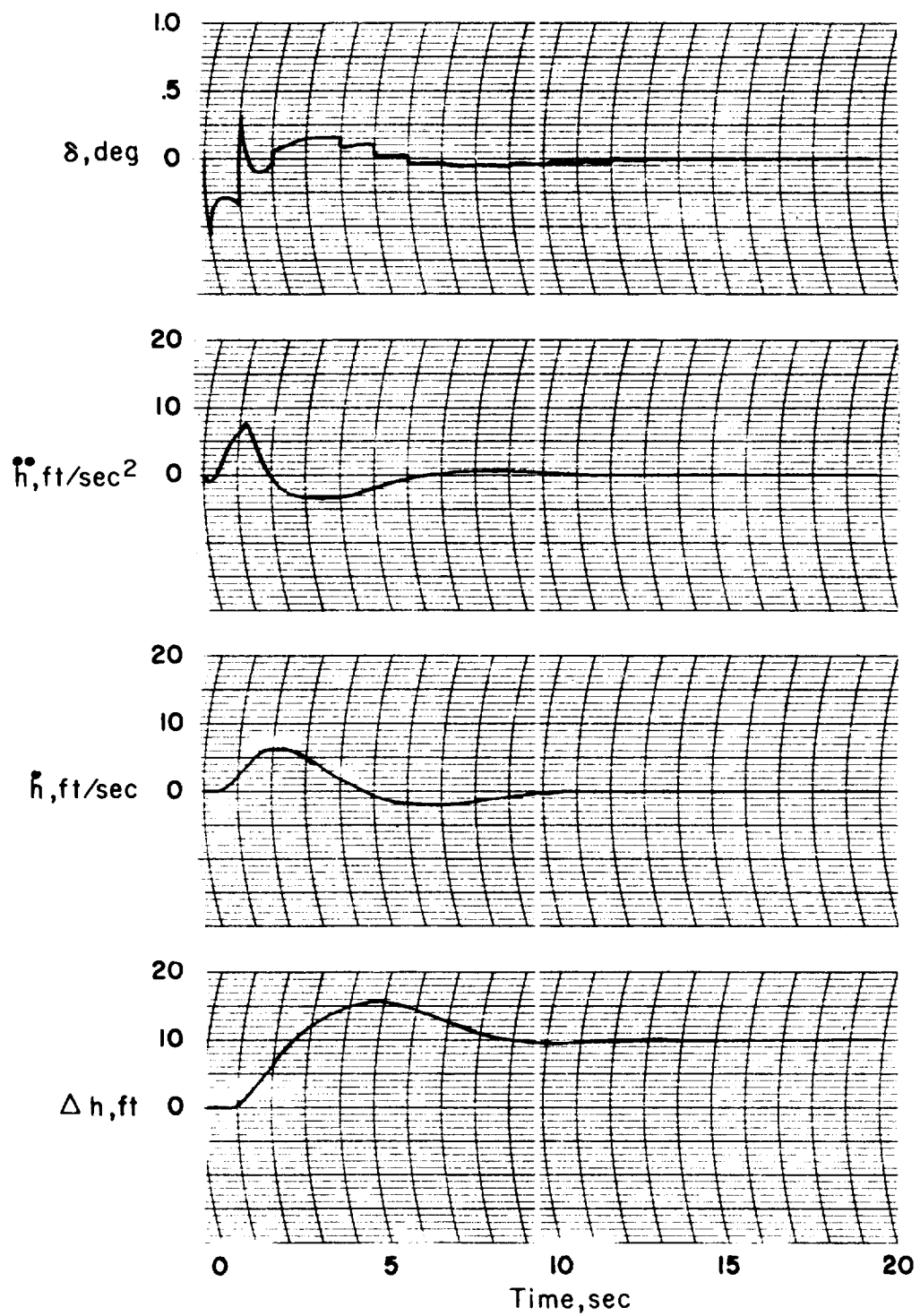
(c) Effects of  $\pm 10$ -percent gain changes; 10-foot step input.

Figure 13.- Continued.



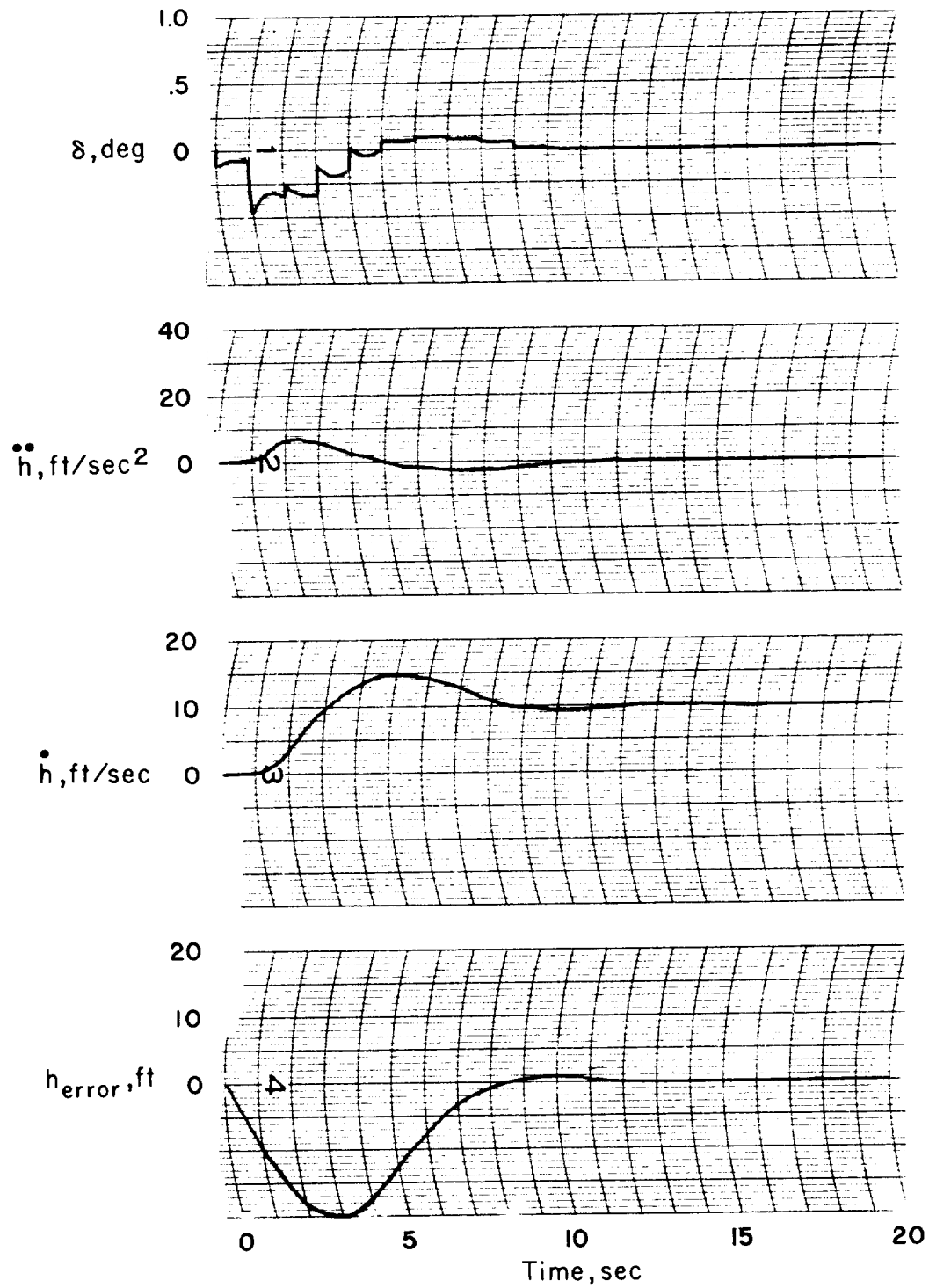
(d) Effects of a rate-limited control-surface servo; 10-foot step input.

Figure 13.- Concluded.



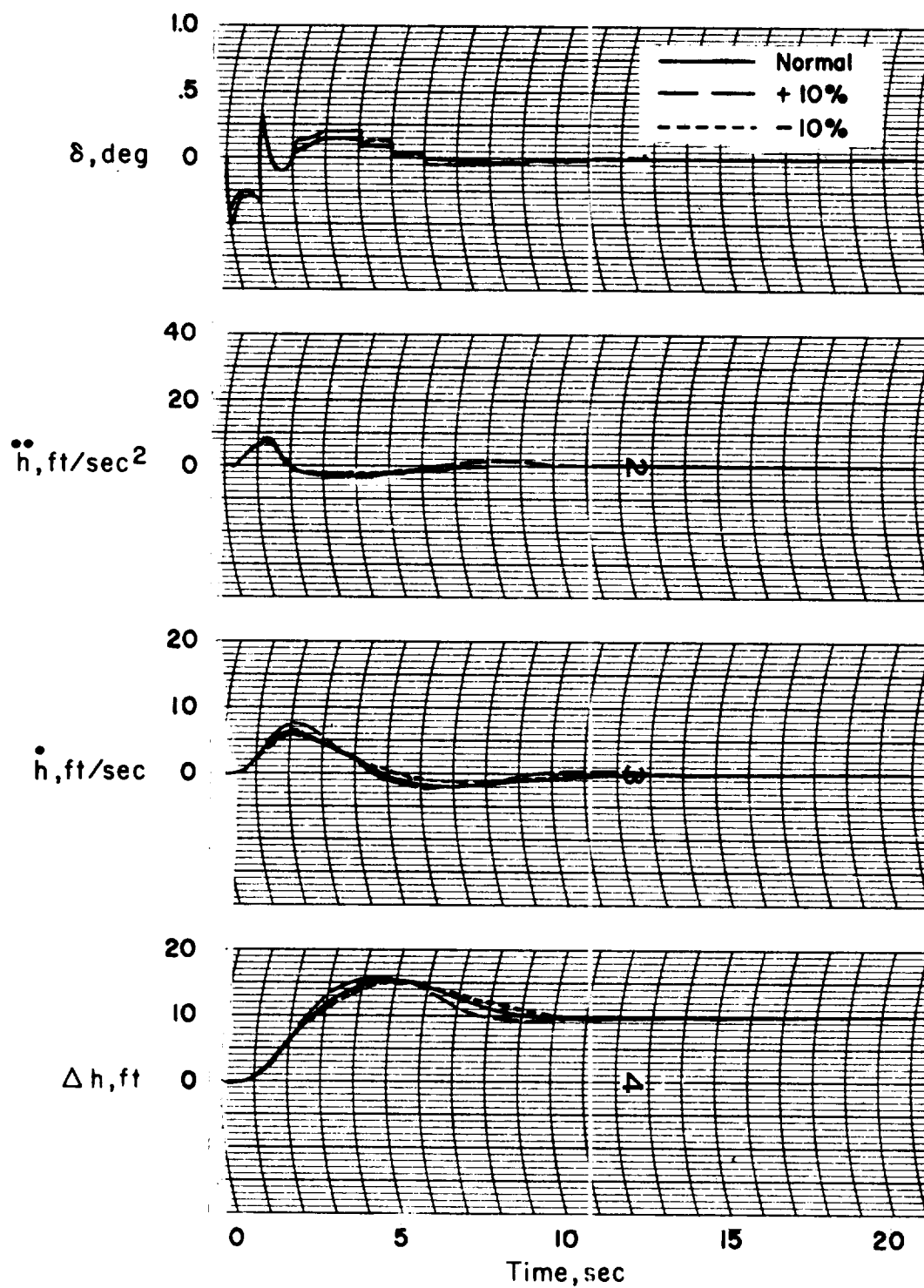
(a) 10-foot step input.

Figure 14.- Transient responses for a modified system, cancellation permitted; Case II,  $\Gamma = 1.0$ .



(b) 10-foot per second ramp input.

Figure 14.- Continued.



(c) Effects of  $\pm 10$  percent gain changes; 10-foot step input.

Figure 14.- Concluded.



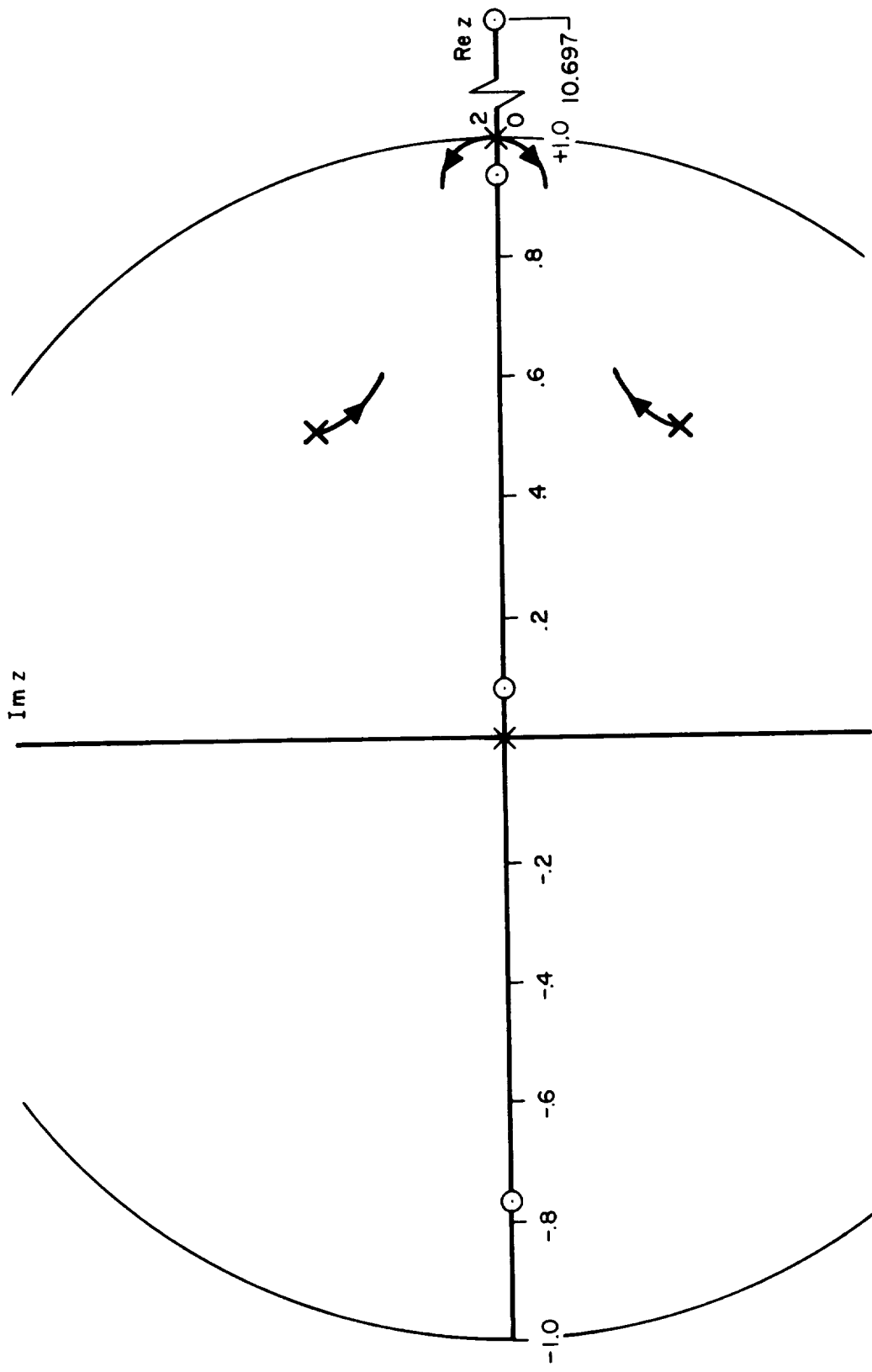
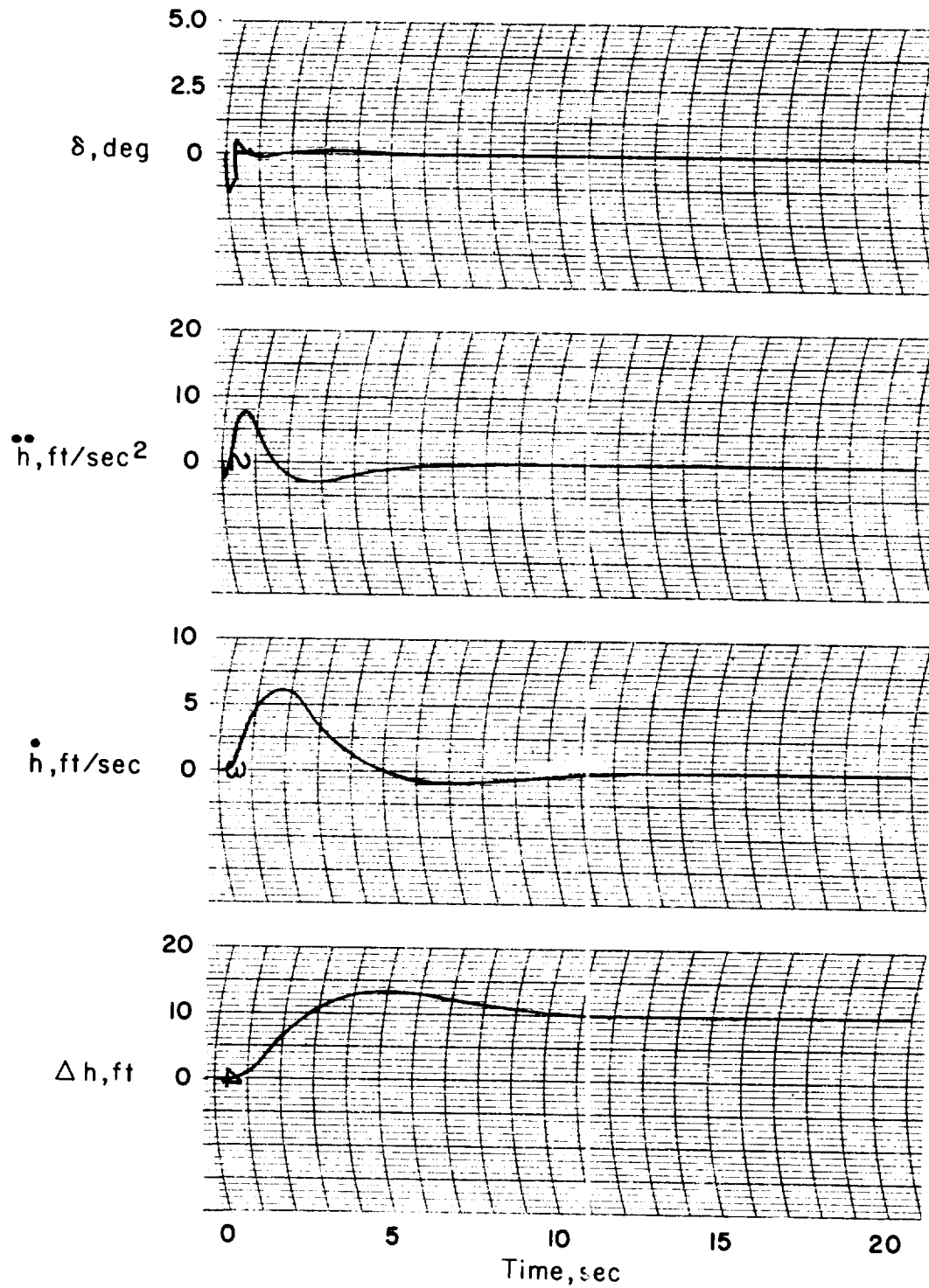
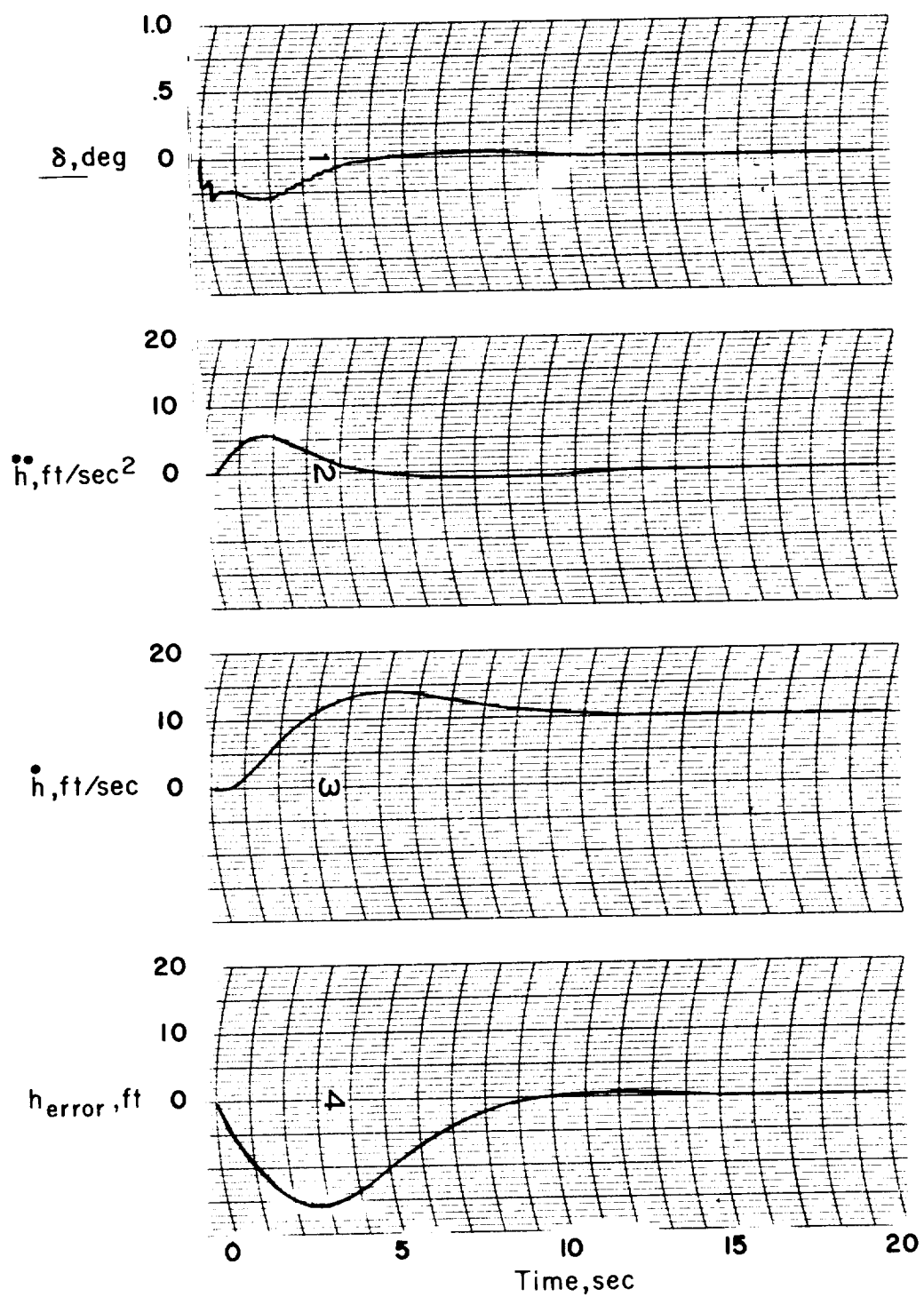


Figure 15.- Root locus for a modified system, cancellation not permitted.



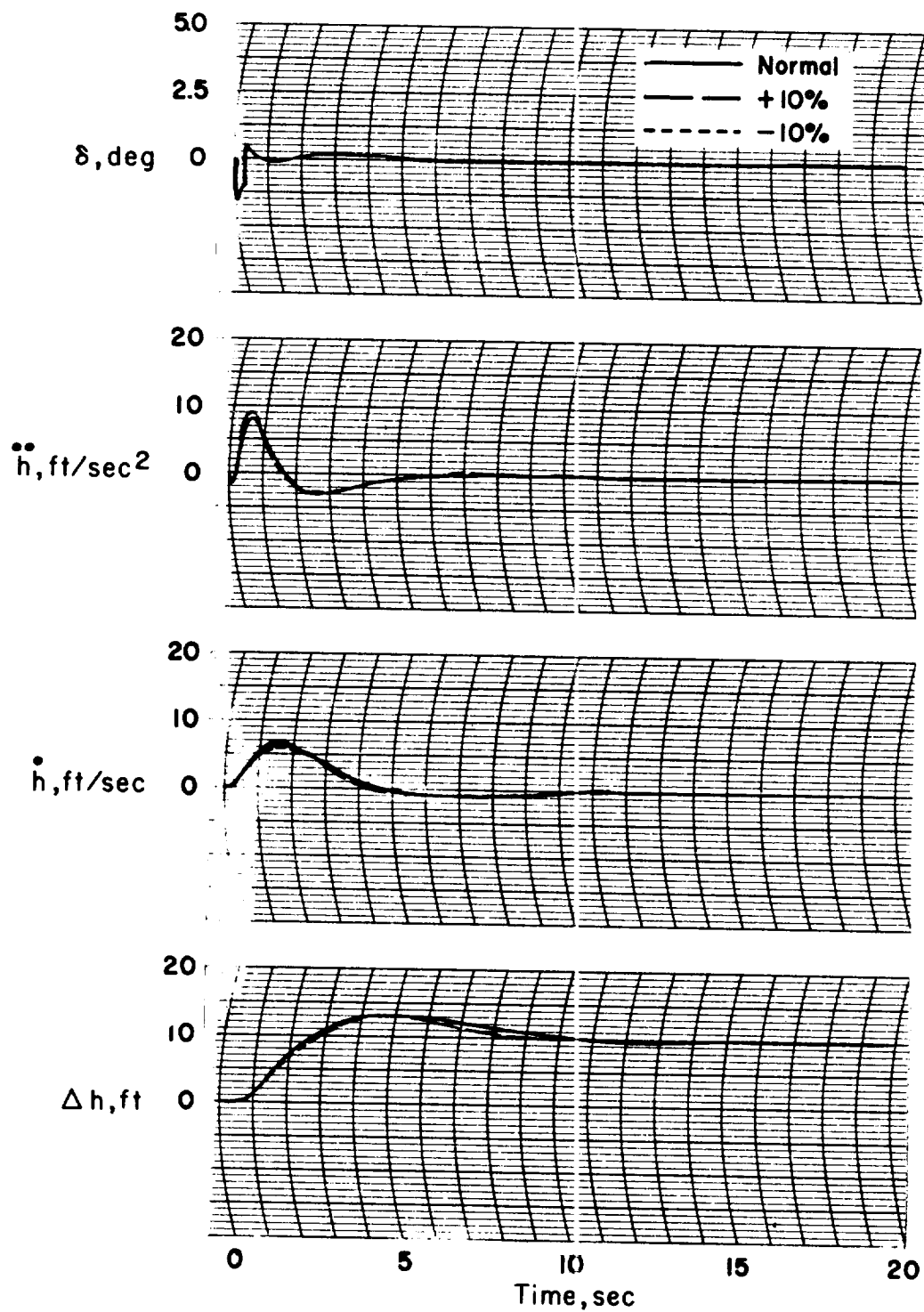
(a) 10-foot step input.

Figure 16.- Transient responses for a modified system, cancellation not permitted; Case II,  $T = 0.25$ .



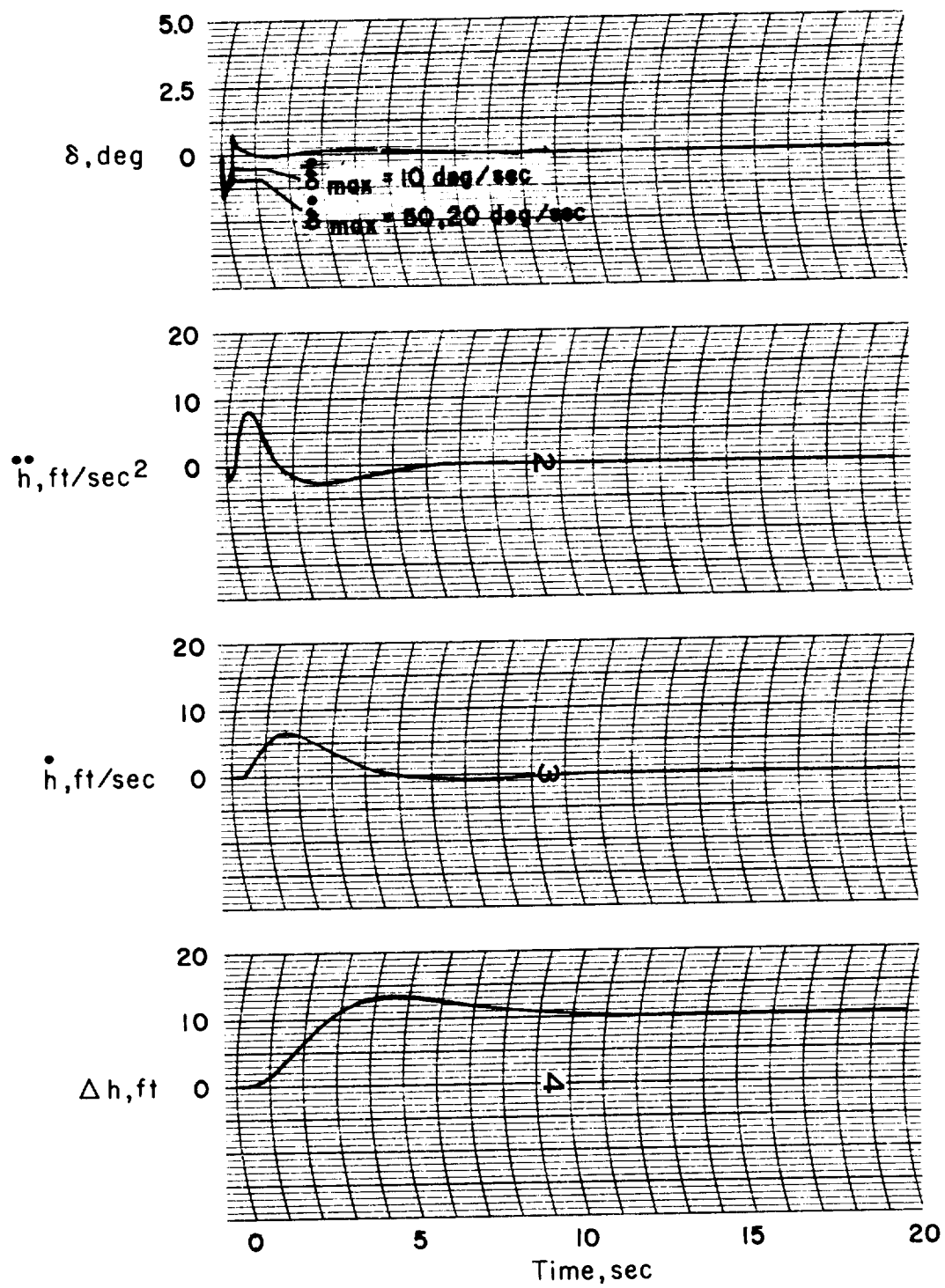
(b) 10-foot per second ramp input.

Figure 16.- Continued.



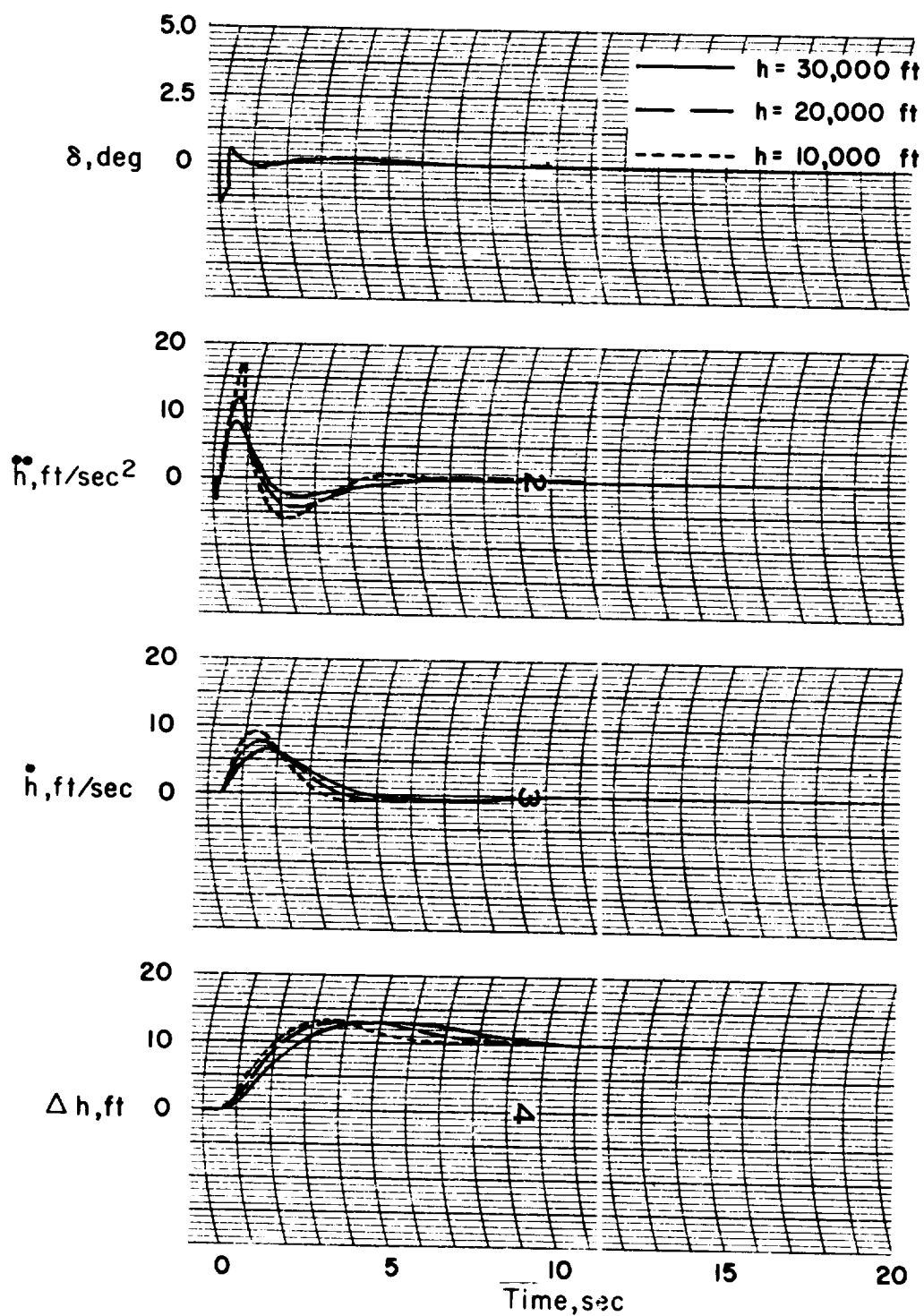
(c) Effects of  $\pm 10$ -percent gain changes; 10-foot step input.

Figure 16.- Continued.



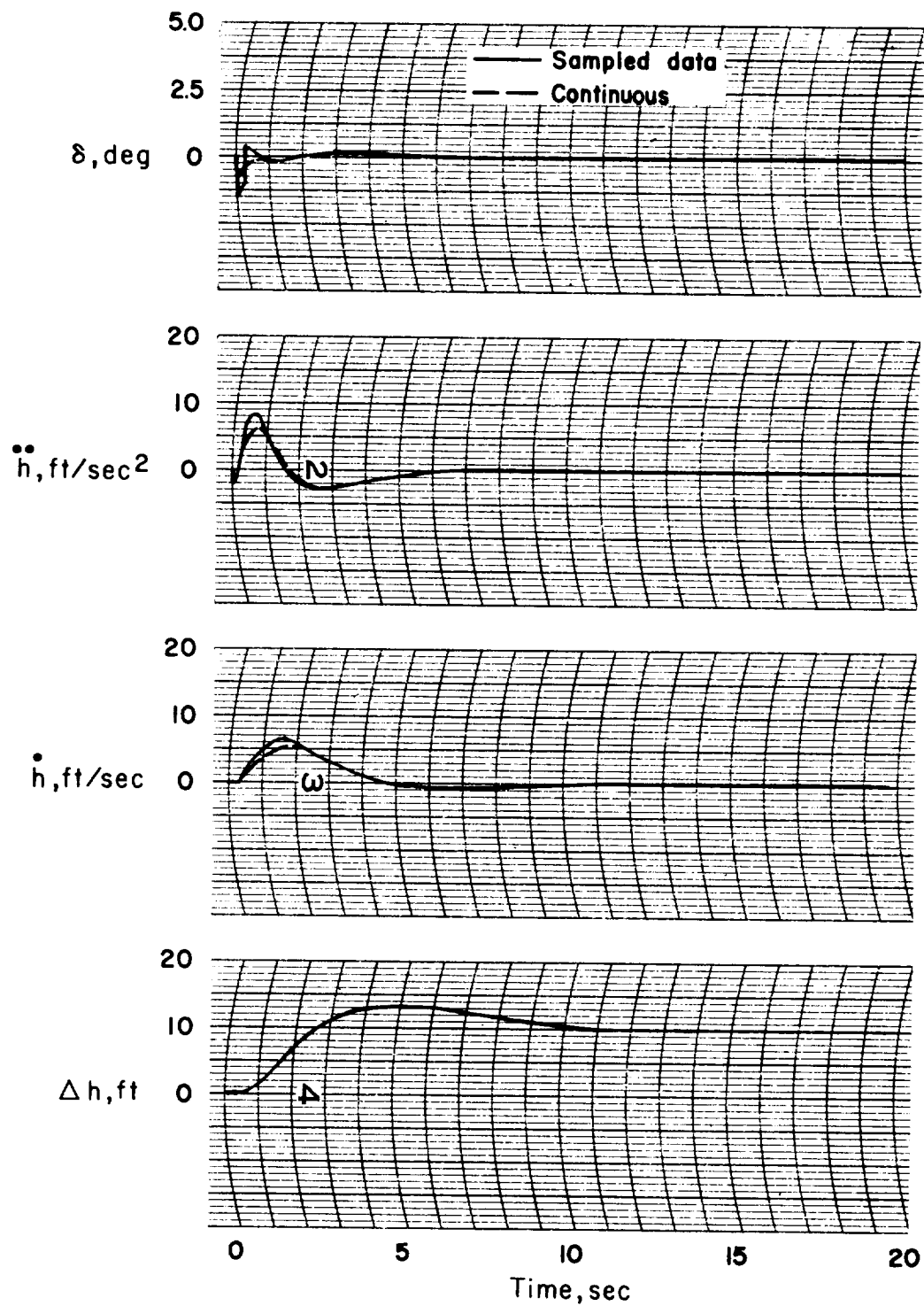
(d) Effects of a rate-limited control-surface servo; 10-foot step input.

Figure 16.- Continued.



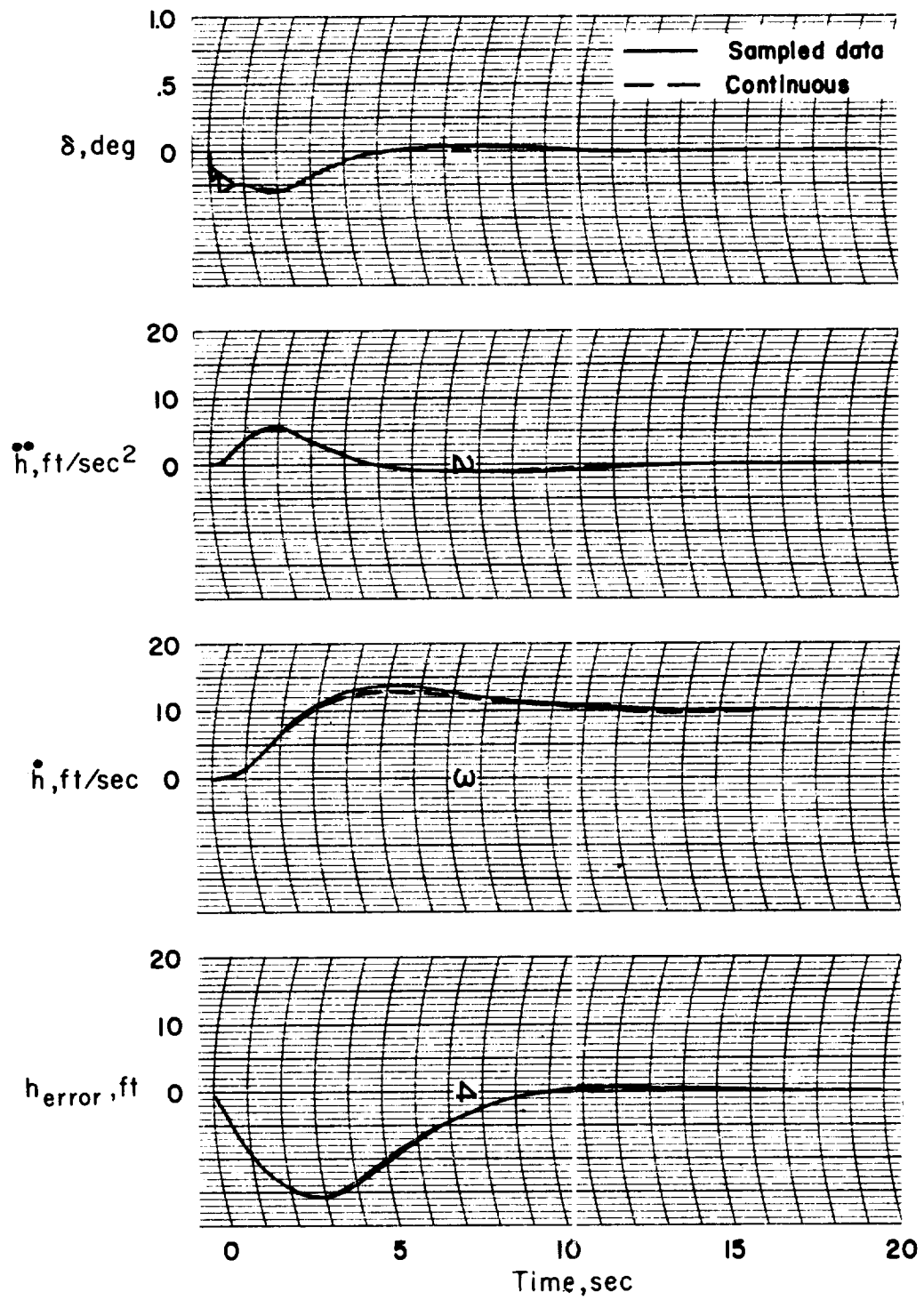
(e) Effects of changes in aircraft altitude; 10-foot step input.

Figure 16.- Concluded.



(a) 10-foot step input.

Figure 17.- Comparison of transient responses of a sampled-data system, cancellation not permitted, with a continuous system.



(b) 10-foot per second ramp input to  $\dot{h}$ .

Figure 17.- Concluded.



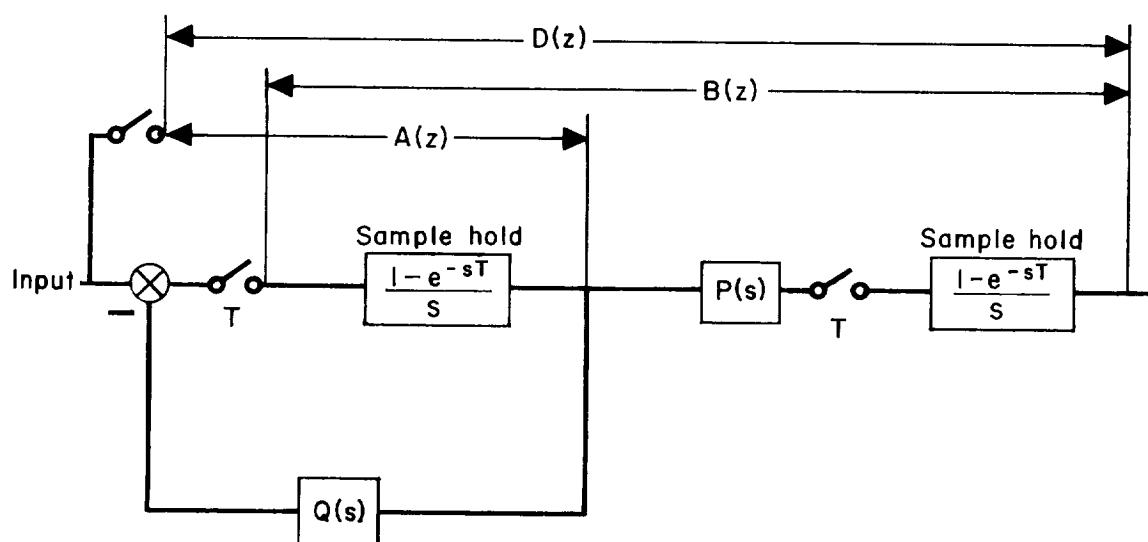


Figure 18.- Block diagram illustrating method of simulating digital-controller pulse transfer function.

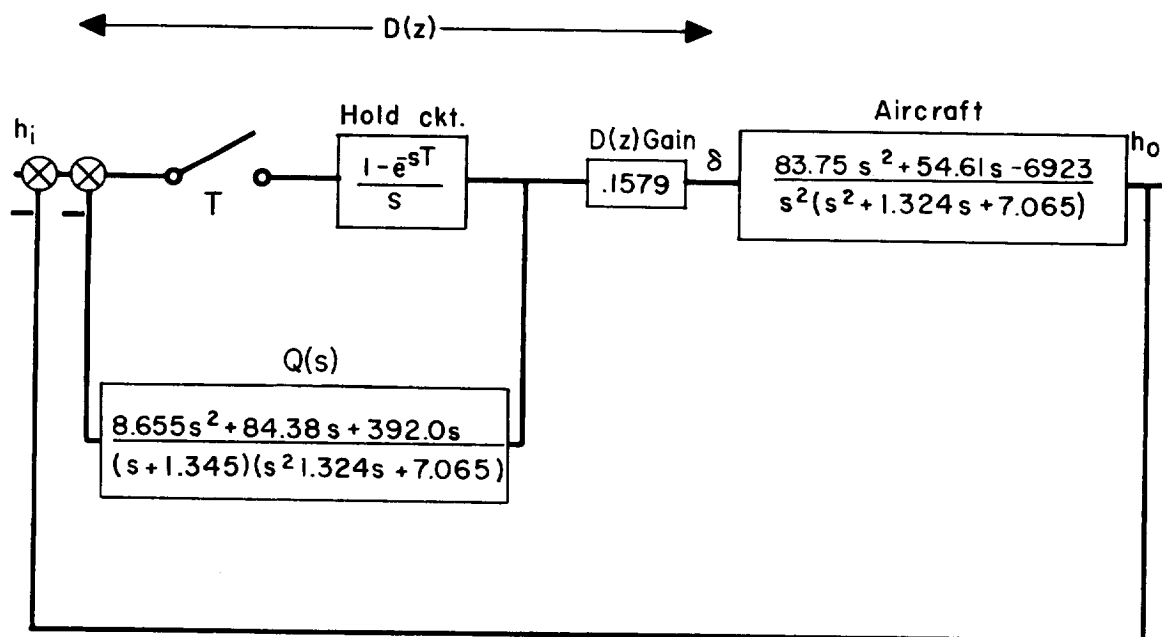


Figure 19.- Block diagram for a finite settling time design; Case I,  $T = 0.25$ .



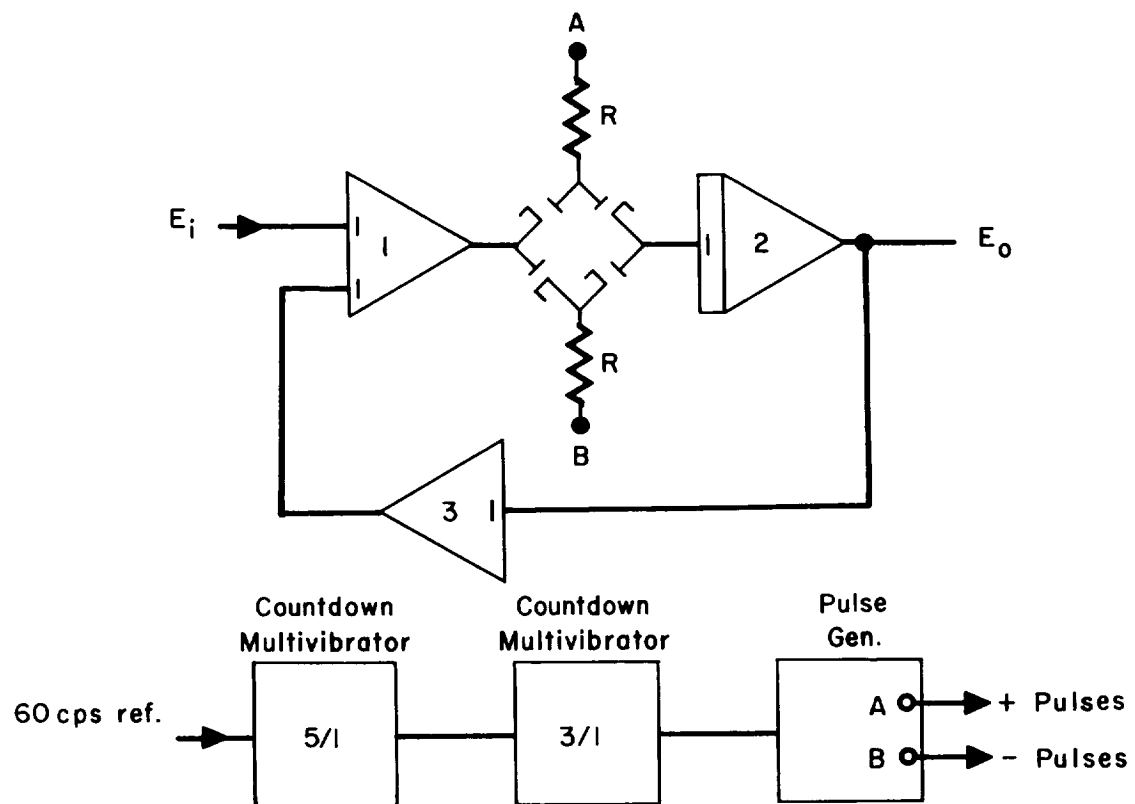


Figure 21.- Block diagram of an electronic sample hold circuit.

